Configurations of Locally Strong Coherence in the Presence of Conditional Exchangeability (the case of cardinality $k \leq 3$)

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Abstract

Locally strong coherence is an helpful property for inference models based on partial lower-upper conditional probabilities. Moreover, structural constraints are usually adopted to improve vague conclusions. In this paper this two aspects are joint together by proposing logical-numerical conditions that guarantee conditional exchangeability among couples or triplets of events.

Keywords. Lower-upper conditional assessments, conditional exchangeability, locally strong coherence.

1 Introduction

In the work started in [10, 13] and recently fully summarized in Coletti & Scozzafava's book [16] logical-numerical conditions that reduce the complexity of automatized procedures based on partial lower-upper conditional probabilities assessments have been introduced. Advantages brought by these conditions relay on a smart use of null probabilities that helps to split a general problem of coherent extension into small subproblems. This procedure transfers most of the computational task from an heavy use of linear optimizations to the detection of particular configurations among the conditional events of the domain. Similar effort has been tackled by other authors [1, 2, 3, 4], while in [19] an approach closer to usual optimization techniques has been presented.

The presence of favorable configurations is checked by solvability of different logical constraints, depending of the values of the numerical assessment.

On the other side, in the last ISIPTA contribution [5] it was shown how a systematic introduction of structural constraints could improve vague results. Among these structural constraints there was the judgement of conditional exchangeability. Conditional exchangeability is particularly apt to express evaluations of symmetries among a group of events conditioned to a common context. For example, it could be adopted whenever there is a pool of experts with similar training assessing their opinions on the same subject conditioned to some possible scenario. Conditional exchangeability can avoid to adopt the stronger assumption of conditional independence that sometimes is not motivated by the problem.

In this paper there is the intention to merge these two notions of locally strong coherence and conditional exchangeability. This is realized by characterizing the subfamilies of the domain that, even under the judgement of conditional exchangeability among their elements, can be considered separately from the rest of the assessment.

The study is limited to subfamilies of cardinality two or three in line with [10] where there is the same limitation for the conditions that have been actually implemented in a specific software.

Obviously, since conditional exchangeability is an additional requirement, the conditions introduced here turn out to be specializations of those already proposed for general assessments.

The rest of the paper is organized as follows. In Section 2 we introduce the basic notions of coherence and conditional exchangeability for partial models. In Section 3 we recall the notion of locally strong coherence and we introduce logical-numerical conditions that guarantee it when a judgment of conditional exchangeability is given on sub-families of cardinality 2 and 3. Their practical application is shown with a simple example. In Section 4 a short concluding remark is given.

2 Preliminaries

2.1 Coherence

Let us briefly introduce the basic notions needed to depict the framework where the proposal will be embedded. A fully detailed and motivated description of all the following concepts can be found in [16].

We deal with problems that can be modeled by partial lower-upper conditional probability assessments. It means that relevant entities are represented by a finite family of conditional events $\mathcal{F} = \{E_i|H_i\}_{i=1}^n$ while uncertainty is expressed trough a vector of numerical bounds $\mathbf{p} = ([lb_1, ub_1], \ldots, [lb_n, ub_n])$. Each closed interval $[lb_i, ub_i]$ represents lower and upper bounds associated with probabilities for the corresponding conditional event $E_i|H_i$. These are usually estimated by expert beliefs, by literature reports or by collected data.

The model is partial because $\mathcal{F}$ describes only the really relevant and available statements, without necessarily reaching to a fully detailed description as usually done by sophisticated statistical models. For this, it is crucial to endow the model with a set of logical constraints $\mathcal{L}_\mathcal{F}$ among the set of unconditional events $\mathcal{U}_\mathcal{F} = \{E_1, H_1, \ldots, E_n, H_n\}$. $\mathcal{L}_\mathcal{F}$ models incompatibilities, implications, coincidences, or whatever, among the events in $\mathcal{U}_\mathcal{F}$ and it can be formalized as a set of logical clauses that in any interpretation coincide with the false event (in the sequel Ω will denote the sure event while ϕ the false one).

As an ancillary tool we need to introduce the set of atoms $\mathcal{A}_\mathcal{F}$ generated by the $\mathcal{U}_\mathcal{F}$ and contained in $\bigvee_{i=1}^n H_i$. Note that $\mathcal{A}_\mathcal{F}$ is not a part of the assessment (i.e. it is not in the "input" of the model) because it is implicitly described by $\mathcal{U}_\mathcal{F}$ and $\mathcal{L}_\mathcal{F}$. Moreover, sometimes we will refer to it (or to some subset of it) to describe theoretical properties even if we will not need to actually generate it.

Elements of $\mathcal{A}_\mathcal{F}$ are obtained combining all situations obtainable by $\mathcal{F}$ and compatible with the given logical constraints $\mathcal{L}_\mathcal{F}$, as shown in the following formula

$$A_r = \bigwedge_{i=1}^n (\widetilde{E_i|H_i}),$$

where $(\widetilde{E_i|H_i})$ can vary among E_iH_i, $E_i^cH_i$, H_i^c (here the conjunction symbol $\wedge$ is omitted, but it will be used whenever the events will have an index varying in an index-set). Be aware that the elementary event $\bigwedge_{i=1}^n H_i^c$ has been excluded from $\mathcal{A}_\mathcal{F}$ because it lives outside every conditioning event H_i (for a more detailed motivation see [20, 24]).

The number of atoms is less or equal than $3^n - 1$. In fact, because of logical relations among the events in $\mathcal{U}_\mathcal{F}$, some combinations turn out to be impossible. The maximum number $3^n - 1$ is reached only when logical constraints are of the form $E_i \subset H_i$ or under the logical independence assumption. Note that, however, the set $\mathcal{A}_\mathcal{F}$ grows exponentially with respect to the given family $\mathcal{F}$.

Since the assessment $(\mathcal{F}, \mathcal{L}_\mathcal{F}, \mathbf{p})$ is partial, it must obey to some consistency rule to be adopted as a reasonable model. We will require $(\mathcal{F}, \mathcal{L}_\mathcal{F}, \mathbf{p})$ to be *coherent*, i.e. that there exist a class $\mathbb{P}_\mathcal{F}$ of conditional probability distributions[1] such that $\mathbf{p}$ coincides with the convex envelope of $\mathbb{P}_\mathcal{F}$ restricted to $\mathcal{F}$, i.e. such that

$$\forall P \in \mathbb{P}_\mathcal{F} \qquad lb_i \leq P(E_i|H_i) \leq ub_i \ \forall E_i|H_i \in \mathcal{F}; \quad (1)$$

$$\forall E_i|H_i \in \mathcal{F} \qquad \exists P', P'' \in \mathbb{P}_\mathcal{F} \text{ s.t. } \begin{array}{l} P'(E_i|H_i) = lb_i \\ P''(E_i|H_i) = ub_i \end{array} \quad (2)$$

Practically speaking, the numerical bounds $\mathbf{p}$ represent a set of numerical constraints that all the admissible models (the conditional probabilities $P \in \mathbb{P}_F$) must satisfy (inequalities (1)). Such constraints must be tight enough so that their bounds can be actually reached by some of the admissible models (equalities (2)).

Note that this coherence notion is almost the same as those usually adopted in imprecise probabilities frameworks. That is, $\mathbf{p}$ coincides with its *natural extension*. (See [25] and [26, §3.2] property (d).)

Also $\mathbb{P}_\mathcal{F}$ is implicitly defined trough the assessment and it is used only potentially to compute, for example, any coherent extension of $\mathbf{p}$ to some inference target.

The check of non-emptiness of $\mathbb{P}_\mathcal{F}$, known as *check of coherence*, is a compulsory step to perform with partial models, especially whenever the information comes from different sources. A real problem connected with such a feature is reported in [6]. In fact, contrary to what could be instinctively thought, problems of coherence can appear especially in the finite context and not only in the continuous case.

[1] Conditional distributions directly defined on Cartesian products were introduced and fully characterized with the following axioms by de Finetti [17] and Dubins [18]:
let $\mathcal{E}$ be a Boolean algebra and $\mathcal{H}$ an additive class, then a function $P : \mathcal{E} \times \mathcal{H} \to [0, 1]$ is a conditional probability distribution if

1. $\forall H \in \mathcal{H}$, $P(\cdot|H) : \mathcal{E} \to [0, 1]$ is an additive probability distribution;
2. $\forall H \in \mathcal{E} \cap \mathcal{H}$, $P(H|H) = 1$;
3. $\forall\ E_1, E_2 \in \mathcal{E}$ and $H \in \mathcal{H}$ s.t. $E_1H \in \mathcal{H}$, $P(E_1E_2|H) = P(E_1|H)\, P(E_2|E_1H)$.

As it has been already stated in [14, 15], and in particular in [16, §15.2], in the finite context the existence of $\mathbb{P}_{\mathcal{F}}$ can be checked *operationally* by the satisfiability of a *class of sequences* of linear systems $\{\mathcal{S}^j_\alpha\}$, with $j = 1, \dots, 2n$ and $\alpha = 1, \dots, \alpha_j$. Note that *sequences* of linear systems are necessary to allow conditioning events H_i's to have induced probabilities that are not bounded away from 0. This procedure partitions $\mathcal{F}$ in different *zero layers* indexed by α. (For a deeper exposition of this aspect refer again to [16], in particular to § 12 and § 15).

The explicit formulation of the linear systems can be simplified by the use of characteristic vectors of the events, i.e. vectors whose components are 1 or 0 depending if the corresponding atom implies or not the event, and we will denote them with the same letter of the event but in boldface lower-cases. Hence, for example, $\mathbf{e_i}$ and $\mathbf{h_i}$ will denote the characteristic vectors of E_i and H_i, respectively, while their juxtaposition $\mathbf{e_i h_i}$ will represent the characteristic vector of the conjunction E_iH_i. Introducing a vector of variables $\mathbf{x} = (x_1 \dots x_a)$, where each component x_r is associated with possible values for the probability of the atom A_r, it is possible to *rebuild* the possible values of probability for any event in $\mathcal{U}_{\mathcal{F}}$, say E_i, simply by

$$P(E_i) = \sum_{A_r \subseteq E_i} P(A_r) = \mathbf{e_i} \cdot \mathbf{x} \ , \tag{3}$$

where $\cdot$ represents the row-column matrix product.

With such a notation linear systems $\{\mathcal{S}^j_\alpha\}$, with $j = 1, \dots, 2n$ and $\alpha = 1, \dots, \alpha_j$, have the following common structure

$$(\mathcal{S}^j_\alpha) \begin{cases} (\mathbf{e_j h_j} - p_j\,\mathbf{h_j}) \cdot \mathbf{x}^\alpha = 0 & \text{if } \mathbf{h_j} \cdot \mathbf{x}^{\alpha-\mathbf{1}} = 0 \\ \\ \begin{array}{l} (\mathbf{e_k h_k} - lb_k\,\mathbf{h_k}) \cdot \mathbf{x}^\alpha \geq 0 \\ (\mathbf{e_k h_k} - ub_k\,\mathbf{h_k}) \cdot \mathbf{x}^\alpha \leq 0 \end{array} & \text{if } \mathbf{h_k} \cdot \mathbf{x}^{\alpha-\mathbf{1}} = 0 \\ \\ \mathbf{x}^\alpha \geq 0 \ , \ \mathbf{x}^\alpha \neq 0 \end{cases} , \tag{4}$$

where $E_j|H_j$ equals $E_i|H_i$ for both subsequent odd and even indexes $j = 2i-1$ and $j = 2i$ (or, conversely $i = \llcorner \frac{j+1}{2} \lrcorner$), k varies among all indexes different from j, while the value p_j that appears in the first equation equals lb_i for the odd indexes $j = 2i - 1$ and ub_i for the even ones $j = 2i$.

Hence, for each event $E_i|H_i \in \mathcal{F}$ there will be associated two sequences of linear systems $\mathcal{S}^{2i-1}_\alpha$ and $\mathcal{S}^{2i}_\alpha$. This to ensure that, according to (2), the bounds lb_i and ub_i can be actually attained. Of course whenever the bounds degenerate to a single value p_i, the two sequences coincide.

In the next sub-section we will see how to introduce at this point the qualitative judgement of conditional exchangeability.

2.2 Conditional exchangeability

As already mentioned, a common method of restricting the variability of the conclusions is to adopt an assumption of stochastic independence. This is actually a powerful restriction, and is not always really appropriate. Specifically, when information is based on judgments made by several experts, the presumed independence of experts is often based on the fact that they each make their judgment without knowing the judgments of the others. But this does not really imply stochastic independence. Stochastic independence of their assessments would mean that we, the probability assessors, would not change our probability assessment for a positive judgment by one expert when we hear the judgment of another expert. This is not really the case, because we explicitly regard them all as experts. What should be modelled is the fact that the judgments are thought to be given in *similar* circumstance and, mainly, by people with the same background. Hence, in the presence of such strong symmetries it is more suitable to introduce some kind of *exchangeability*. (For another similar situation, refer to Lad and Di Bacco [22].)

In fact exchangeability reflects information of perfect permutability among a set of events, that usually represent judgments or experiments, and it is appropriate whenever it is relevant to consider *how many* instead of *which particular* events hold.

More technically, exchangeability should be used whenever it is possible to identify a *sum* as a sufficient statistic [2] (for a detailed explanation refer to [21, §3.9]).

In particular, whenever the assessment is mainly conditional, the judgement of *conditional exchangeability* could be the more suitable tool to adopt and it is formulated as follows:

Definition 1 *k events $E_1, \dots, E_k$ are regarded as exchangeable* under a specific scenario H_l *if any conjunction of the E_i's with the same number of affirmed and negated events is evaluated identically when conditioned upon H_l. In other words, for any fixed number $s \in \{0, \dots, k\}$ the probabilities*

$$P(E_{i_1} \dots E_{i_s} \neg E_{i_{s+1}} \dots \neg E_{i_k} | H_l) \tag{5}$$

are assessed to be equal for any permutation of the indexes $i_1, \dots, i_k$.

[2]Exchangeability is not the only property that takes into account a sum as a relevant quantity [23] but it is the most natural to adopt whenever there is a fully symmetric judgment among the events.

Conditions like (5) actually reduce the "degree of freedom" for the unknowns in the sequences of linear systems for the check of coherence. This restricts "de facto" the admissible class of conditional measures $\mathbb{P}_{\mathcal{F}}$.

Since (5) refers to a fixed conditioning event H_l, restriction of this type are easily reported as linear constraints. In fact, let us denote with $\boldsymbol{\pi_s}$ and $\boldsymbol{\pi'_s}$ the characteristic vectors of two different permutations of the combination $E_{i_1} \ldots E_{i_s} \neg E_{i_{s+1}} \ldots \neg E_{i_k}$ and with $\mathbf{x}^\alpha$ a generic vector of variables of the j-th sequence of linear systems. Hence coherence with the further conditional exchangeability requirement (5) follow by adding to the constraints of the linear systems (4) pairwise equalities of the form

$$(\boldsymbol{\pi_s h_l} - \boldsymbol{\pi'_s h_l}) \cdot \mathbf{x}^\alpha = 0 \tag{6}$$

for each pair of permutations $\boldsymbol{\pi_s}$ and $\boldsymbol{\pi'_s}$ and each $s = 1, \ldots, k-1$. (Note that the extreme cases $s = 0$ and $s = k$ do not actually constitute any constraint, because only one arrangement of "all 1's" or "all 0's" is possible).

3 Locally strong coherence

3.1 General setting

Till now we have referred to a direct involvement of the linear systems $\mathcal{S}^j_\alpha$. Nevertheless, $\mathcal{S}^j_0$ has an exponential number of unknowns with respect to the number of events in $\mathcal{F}$. So, the large number of unknowns could make the problem not manageable from a computational point of view.

However, Coletti et al. [13] showed it is possible, with the help of *zero probabilities*, to make easier the procedure trying to solve "smaller systems", and building only some atoms. This is possible by a careful choice of the conditioning events whose probability can be put equal to zero.

This idea of exploiting zero probabilities has been used [9, 10] to characterize *locally strong coherence*. It was adopted such a name because the notion is *stronger* than usual coherence since it *forces* some of the conditional events in $\mathcal{F}$ to lie in the same zero layer, moreover it involve subfamilies of $\mathcal{F}$ and hence it is *locally* checked.

This concept, whose formal properties relay anyway on linear constraints satisfaction, has the good aspect to be operationally characterized by the satisfiability of logical-numerical conditions.

In the following we report the formal definition of locally strong coherence to better understand the new logical-numerical properties we will introduce for conditional exchangeability.

Let now $\mathcal{G}$ be a subfamily of $\mathcal{F}$, we denote by $\mathcal{R} = \mathcal{F} \setminus \mathcal{G}$ the remaining elements of the domain and we put

$$\mathcal{H}^c_{\mathcal{R}} = \left(\bigvee_{r:\, E_r|H_r \in \mathcal{R}} H_r \right)^c = \bigwedge_{r:\, E_r|H_r \in \mathcal{R}} H^c_r .$$

Practically speaking, $\mathcal{H}^c_{\mathcal{R}}$ represents all the circumstances not involved by $\mathcal{R}$. In fact, any atom in $\mathcal{H}^c_{\mathcal{R}}$ would falsify all the premises H_r's of the elements of $\mathcal{R}$.

Note that, since the procedure to check the coherence of $(\mathcal{F}, \mathcal{L}_{\mathcal{F}}, \mathbf{p})$ is divided among the different $2n$ sequences, each one characterized by a particular equality in one of the constraints, the notion of locally strong coherence $\mathbf{p}$ must be *specified* to any particular j-th sequence. This aspect differs from what we have for precise assessments where the property of locally strong coherence has always a global effect. In fact for precise assessments the following relation between locally strong coherence and global coherence holds [9]:

Proposition 1 *Let $P : \mathcal{F} \longrightarrow [0, 1]$ be a precise conditional probability assessment locally strong coherent on $\mathcal{G}$. Then P is coherent if and only if its restriction $P_{|\mathcal{R}}$ is coherent.*

For imprecise assessments locally strong coherence can be *specialized* to any one of the $2n$ sequences so that, whenever it subsists, the sequence can be shortened by deleting from the original assessment the subfamily $\mathcal{G}$. This can be formalized with the following definition:

Definition 2 *The assessment $(\mathcal{F}, \mathcal{L}_{\mathcal{F}}, \mathbf{p})$ is partial* locally strong coherent *on $\mathcal{G} \subseteq \mathcal{F}$ with respect the j-th sequence if there exists an unconditional probability P_0 such that*

$$\begin{array}{rcll} P_0(H_r) & = & 0 & \forall H_r \in \mathcal{H}_{\mathcal{R}} \\ \dfrac{P_0(E_j H_j)}{P_0(H_j)} & = & p_j & \textit{if } E_j|H_j \in \mathcal{G} \\ lb_k \leq & \frac{P_0(E_k H_k)}{P_0(H_k)} & \leq ub_k & \forall E_k|H_k \in \mathcal{G} \setminus \{E_j|H_j\}. \end{array} \tag{7}$$

where, again, $E_j|H_j$ equals $E_i|H_i$ for both subsequent odd and even indexes $j = 2i - 1$ and $j = 2i$ while p_j equals lb_i for $j = 2i - 1$ and ub_i for $j = 2i$.

It's immediate to see that the checking of partial locally strong coherence on $\mathcal{G}$ with respect the j-th sequence is equivalent to check the satisfiability of the following linear system

$$\left(S_{\mathcal{G}}^{j}\right)\begin{cases}\mathbf{h_r}\cdot\mathbf{x^0}=0 & \forall H_r\in\mathcal{H}_{\mathcal{R}}\\[1ex] (\mathbf{e_j h_j}-p_j\,\mathbf{h_j})\cdot\mathbf{x^0}=0 & \\ \mathbf{h_j}\cdot\mathbf{x^0}>0 & \\[1ex] (\mathbf{e_k h_k}-lb_k\,\mathbf{h_k})\cdot\mathbf{x^0}\geq 0 & \\ (\mathbf{e_k h_k}-ub_k\,\mathbf{h_k})\cdot\mathbf{x^0}\leq 0 & \forall E_k|H_k\in\mathcal{G}\backslash\{E_j|H_j\}\\ \mathbf{h_k}\cdot\mathbf{x^0}>0 & \\[1ex] \mathbf{x^0}\geq 0 & \end{cases}. \tag{8}$$

Note the equality constraint for the index j, characteristic of the j-th sequence.

Expression (8) reflects the operational relevance of the locally strong coherence. In fact system $S_{\mathcal{G}}^{j}$ can substitute the first element $\mathcal{S}_0^j$ of the sequence, so that the construction of the linear systems can be limited to the constraints related to $\mathcal{R}$. This can be obviously iterated till some favorable sub-family is detected, so that linear systems $\mathcal{S}_\alpha^j$ are actually used only if the domain cannot be reduced anymore.

Obviously, till the partial locally coherence is characterized by linear systems (8) there isn't any real computational benefit. Anyhow, as it has been already shown in [9, 10], the use of linear systems like (8) can be "bypassed" by testing the satisfiability of particular configurations among the events of $\mathcal{G}$ relative to $\mathcal{H}_{\mathcal{R}}^c$. Such logical configurations are classified with respect to the cardinality k of the sub-families $\mathcal{G}$ and to the conditional probabilities bounds $[lb_i, ub_i]$, $E_i|H_i\in\mathcal{G}$. We can now state similar conditions whenever we take as subfamily $\mathcal{G}=\{E_1|H_l,\ldots,E_k|H_l\}$ a set of conditionally exchangeable events with cardinality $k=2$ or $k=3$.

3.2 Logical-numerical conditions for locally strong coherence on sub-families of conditional exchangeable events with cardinality 2 and 3

The additional requirement of conditional exchangeability among the elements of the sub-family $\mathcal{G}=\{E_1|H_l,\ldots,E_k|H_l\}$ reflects on peculiar properties for the linear systems whose solution would be guaranteed by favorable configurations. Note in fact that (5) implies equiprobability among the conditional events $E_i|H_l\in\mathcal{G}$, while, from (7) and since in $\mathcal{G}$ there is a common conditioning event H_l, it is possible to impose the further constraint

$$\mathbf{h_l}\cdot\mathbf{x^0}=1 \tag{9}$$

without loss of generality.

For these reasons the numerical bounds $[lb_i, ub_i]$ must coincide for all $E_i|H_l$ in $\mathcal{G}$, the $2k$ sequences associated to the elements of $\mathcal{G}$ reduce only to one referred to lb_i and an other to ub_i, while linear systems (8) reduce to the special form

$$\begin{cases}\mathbf{h_r}\cdot\mathbf{x^0} & = & 0 & \forall H_r\in\mathcal{H}_{\mathcal{R}}\\[1ex] \mathbf{e_i h_l}\cdot\mathbf{x^0}-p_i & = & 0 & \forall E_i|H_l\in\mathcal{G}\\[1ex] \mathbf{h_l}\cdot\mathbf{x^0} & = & 1 & \\[1ex] \mathbf{x^0}\geq 0 & & & \end{cases} \tag{10}$$

with respect the sequences where one of the extremes $p_i=lb_i$ or $p_i=ub_i$ must be attained, or

$$\begin{cases}\mathbf{h_r}\cdot\mathbf{x^0}=0 & \forall H_r\in\mathcal{H}_{\mathcal{R}}\\[1ex] \mathbf{e_i h_l}\cdot\mathbf{x^0}-lb_i\geq 0 & \\ \mathbf{e_i h_l}\cdot\mathbf{x^0}-ub_i\leq 0 & \forall E_i|H_l\in\mathcal{G}\\[1ex] \mathbf{h_l}\cdot\mathbf{x^0}=1 & \\[1ex] \mathbf{x^0}\geq 0 & \end{cases} \tag{11}$$

with respect to the other sequences.

The problem now is to identify logical configurations, i.e. not impossible atoms, among the elements of $\mathcal{G}$ that ensure the existence of solutions of linear systems like (10) or like (11).

For sub-families $\mathcal{G}$ with cardinality $k=2$ favorable configurations can be picked out nothing that by (5) it is possible to express the conditional probability of any element of $\mathcal{G}$ at the first zero-layer $\alpha=0$ by

$$\frac{P_0(E_iH_l)}{P_0(H_l)}=\frac{\varrho(1)+\varrho(2)}{\varrho(0)+2\varrho(1)+\varrho(2)} \tag{12}$$

where $\varrho(0)=P_0(E_1^cE_2^cH_l\mathcal{H}_{\mathcal{R}}^c)$, $\varrho(1)=P_0(E_1E_2^cH_l\mathcal{H}_{\mathcal{R}}^c)=P_0(E_1^cE_2H_l\mathcal{H}_{\mathcal{R}}^c)$ and $\varrho(2)=P_0(E_1E_2H_l\mathcal{H}_{\mathcal{R}}^c)$. Moreover, by the normalization constraint (9), it follows that it should be possible to find not negative values for the $\varrho(\cdot)$ such that

$$\begin{cases}\varrho(1)+\varrho(2) & = & b_i\\ \varrho(0)+2\varrho(1)+\varrho(2) & = & 1\end{cases} \tag{13}$$

where now b_i can be either one of the extreme values $p_i\in\{lb_i, ub_i\}$ for the sequences of applicability of (10) or any value inside the interval $[lb_i, ub_i]$ in the other cases of applicability of (11).

Favorable configurations are the atoms associated to the $\varrho(\cdot)$ that must be *strictly positive* to have a solution of (13), hence they will depend on the numerical value b_i. A full classification is shown in Tab. 1 and it generalizes the similar taxonomy reported in [11] for precise assessments. The alternative favorable configurations are explicitly reported as the atoms that cannot be impossible and they are listed according to ranges of values admissible for b_i.

b_i range	favorable configurations ♯1	♯2
0	$E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c$	-
$(0,1/2)$	$\left\{\begin{array}{l} E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$1/2$	$\left\{\begin{array}{l} E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$(1/2,1)$	$\left\{\begin{array}{l} E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1E_2H_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
1	$E_1E_2H_l\mathcal{H}_\mathcal{R}^c$	-

Table 1: Lists of atoms that guarantee the partial locally strong coherence of $\mathcal{G} = \{E_1|H_l, E_2|H_l\}$ with respect an appropriate sequence depending on the value of $b_i \in \{lb_i, ub_i\}$ or $b_i \in [lb_i, ub_i]$.

Obviously, whenever the interval $[lb_i, ub_i]$ overlaps different ranges of values, the existence of *at least* one of the corresponding favorable conditions guarantees the partial locally strong coherence.

Note that the configuration

$$\left\{\begin{array}{l} E_1^cE_2^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2H_l\mathcal{H}_\mathcal{R}^c \end{array}\right. \tag{14}$$

guarantees the locally strong coherence irrespectively of the value of b_i, hence if it holds there is a *global* effect so that the sub-family $\mathcal{G}$ can be deleted from every j-th sequence, $j = 1, \ldots, 2n$.

For sub-families $\mathcal{G}$ of cardinality $k = 3$, i.e. $\mathcal{G} = \{E_1|H_l, E_2|H_l, E_3|H_l\}$, the reasoning is the same, with the difference that (13) modifies in

$$\left\{\begin{array}{lcl} \varrho(1) + 2\varrho(2) + \varrho(3) & = & b_i \\ \varrho(0) + 3\varrho(1) + 3\varrho(2) + \varrho(3) & = & 1 \end{array}\right. \tag{15}$$

where now

$$\begin{array}{rcl} \varrho(0) & = & P_0(E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c) \\ \varrho(1) & = & P_0(E_1E_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c) = \\ & = & P_0(E_1^cE_2E_3^cH_l\mathcal{H}_\mathcal{R}^c) = \\ & = & P_0(E_1^cE_2^cE_3H_l\mathcal{H}_\mathcal{R}^c) \\ \varrho(2) & = & P_0(E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c) = \\ & = & P_0(E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c) = \\ & = & P_0(E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c) \\ \varrho(3) & = & P_0(E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c)\ . \end{array}$$

Due to space requirements, the full classification of favorable configurations is reported on the two tables Tab. 2 and Tab. 3. Once again the list of atoms that ensure the locally strong coherence depend on which range the value b_i can varies and in Tab. 3 only ranges that admit more than two favorable configurations have been reported.

b_i range	favorable configurations ♯1	♯2
0	$E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c$	-
$(0,1/3)$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$1/3$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$(1/3,2/3)$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$2/3$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
$(2/3,1)$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$	$\left\{\begin{array}{l} E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right.$
1	$E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c$	-

Table 2: First list of configurations that guarantee the partial locally strong coherence of $\mathcal{G} = \{E_1|H_l, E_2|H_l, E_3|H_l\}$ with respect an appropriate sequence depending on the value of $b_i \in \{lb_i, ub_i\}$ or $b_i \in [lb_i, ub_i]$.

Note that there is again a favorable configuration

$$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3H_l\mathcal{H}_\mathcal{R}^c \end{array}\right. \tag{16}$$

shared by all the possible ranges, hence if it is present it has a *global* effect independently from the numerical values of $[lb_i, ub_i]$, while there are other two cases that have a global effect for more specific situation, i.e.

$$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}_\mathcal{R}^c \\ E_1^cE_2E_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2^cE_3H_l\mathcal{H}_\mathcal{R}^c \\ E_1E_2E_3^cH_l\mathcal{H}_\mathcal{R}^c \end{array}\right. \tag{17}$$

guarantees the locally strong coherence of $\mathcal{G}$ with respect every sequence whenever $[lb_i, ub_i] \subseteq [0, 2/3]$, while

$$\left\{\begin{array}{l} E_1E_2E_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right. \tag{18}$$

has a global effect whenever $[lb_i, ub_i] \subseteq [1/3, 1]$.

b_i range	favorable configurations ♯3	favorable configurations ♯4
$(0, 1/3)$	$\left\{\begin{array}{l} E_1^cE_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$	-
$1/3$	$\left\{\begin{array}{l} E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$	-
$(1/3, 2/3)$	$\left\{\begin{array}{l} E_1E_2E_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$	$\left\{\begin{array}{l} E_1^cE_2E_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$
$2/3$	$\left\{\begin{array}{l} E_1E_2E_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$	-
$(2/3, 1)$	$\left\{\begin{array}{l} E_1E_2E_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2^cE_3H_l\mathcal{H}^c_{\mathcal{R}} \\ E_1E_2^cE_3^cH_l\mathcal{H}^c_{\mathcal{R}} \\ E_1^cE_2E_3^cH_l\mathcal{H}^c_{\mathcal{R}} \end{array}\right.$	-

Table 3: Further configurations that guarantee the partial locally strong coherence of $\mathcal{G} = \{E_1|H_l, E_2|H_l, E_3|H_l\}$.

Hence we have that the original expression of the property of locally strong coherence through satisfiability of linear systems (10) and (11) can be substituted with a problem favorable configurations detection. The presence or not in the model of the favorable configurations listed before, both for the case $k = 2$ and $k = 3$, are operationally tested by checking the logical satisfiability of them jointly with the logical formulae that represent $\mathcal{L}_{\mathcal{F}}$.

Let us conclude this section by showing a simple example of application of these logical-numerical conditions for the locally strong coherence. The assessment in the example is extremely simple because it intends just to show the applicability of the proposed procedure. A more detailed example could distract the reader from the proposal.

Example 3.1 *In [5] it was shown benefits of conditional exchangeability applied to a diagnosis procedure for asbestos induced fibrosis based on a median decision criterion. In that example the numerical assessment was made by precise conditional probabilities, anyhow it was just one of different assessments contemplated on the original work reported in [7]. If we merge all the precise evaluations in a single lower-upper conditional probabilities assessment we obtain the following model:*

$$\mathcal{F} = \{D_1|F, D_2|F, D_3|F, D^*, S^*|D^*\} \ ;$$

where

label	description
F	*asbestosis (*fibrosis*) presence*
D_i	*i-th expert positive asbestosis judgment* $i = 1, 2, 3$
D^*	*positive median decision diagnosis*
S^*	*positive median decision with a splitting vote*

while the other components of the assessment are

$$\mathbf{p} = \left\{\begin{array}{ll} P(D_i|F) \in [.82, .96] & i = 1, 2, 3 \\ P(D^*) = .12 & \\ P(S^*|D^*) = .42 & \end{array}\right. \quad ;$$

and

$$\mathcal{L}_{\mathcal{F}} = \left\{\begin{array}{l} S^* \equiv (D_1D_2D_3^c) \vee (D_1D_2^cD_3) \vee (D_1^cD_2D_3) \\ D^* \equiv S^* \vee (D_1D_2D_3) \end{array}\right.$$

with the further judgement of the D_i, $i = 1, 2, 3$, being exchangeable conditionally on F.

Since on the subfamily $\mathcal{G}_1 = \{D^*, S^*|D^*\}$ the numerical assessment is precise, $\mathcal{G}_1$ can be eliminated by $\mathcal{F}$ because one of the condition for the locally strong coherence reported in [9] is satisfied. In fact, since $0 < P(D^*) = .12 < P(S^*|D^*) = .42 < 1$ and

$$\left\{\begin{array}{rcl} D^*\Omega S^*D^*F^c \equiv S^*F^c & \neq & \phi \\ S^{*c}D^*F^c & \neq & \phi \\ D^{*c}\Omega D^{*c}F^c \equiv D^{*c}F^c & \neq & \phi \end{array}\right. ,$$

then condition (g1b) *in [9] can be applied and the domain reduced to $\mathcal{R}_1 = \{D_1|F, D_2|F, D_3|F\}$.*

Now, the elements of $\mathcal{R}_1$ are judged conditionally exchangeable, and on it we have the lower-upper assessment $[lb_i, ub_i] = [.82, .96]$, $i = 1, 2, 3$.

Moreover, by the logical constraints $\mathcal{L}_{\mathcal{F}}$ it follows that the configuration

$$\left\{\begin{array}{l} D_1D_2D_3F \\ D_1^cD_2^cD_3^cF \end{array}\right.$$

is possible. Hence, by (16), we can ignore $\mathcal{R}_1$, concluding that the original whole assessment $(\mathcal{F}, \mathcal{L}_{\mathcal{F}}, \mathbf{p})$ is coherent. Note that we have obtained this result only through the locally strong coherence without the use of any linear system.

4 Concluding remarks

In the present contribution it was shown how to operationally include the qualitative conditions of conditional exchangeability in models based on partial lower-upper conditional probability assessments. Thanks to the detection of particular favorable configurations among the elements of the domain it is in fact possible to reduce the use of massive linear systems. This was limited to cases where the events judged conditionally exchangeable are 2 or 3. Even being of small dimensions, such cases constitute a valid base of applicability for the methodology. Moreover these are the same cardinalities of the configurations implemented for models that profit from the generic locally strong coherence. Hence the proposed classification can be easily adjoint in the next future to the already existing software engine[3].

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