Answers to Two Questions of Fishburn on Subset Comparisons in Comparative Probability Orderings

Robin Christian
Department of Mathematics
The University of Auckland
rchr019@math.auckland.ac.nz

Arkadii Slinko
Department of Mathematics
The University of Auckland
slinko@math.auckland.ac.nz

Abstract

We show that every additively representable comparative probability ordering is determined by at least $n-1$ binary subset comparisons. We show that there are many orderings of this kind, not just the lexicographic ordering. These results provide answers to two questions asked by Fishburn.

Keywords. Additively representable linear orders, comparative probability, subjective probability, subset comparisons

1 Introduction

Let $\mathcal{L}_n$ be the set of all comparative probability orderings on the set consisting of n elements which are representable by an order preserving positive measure. Fishburn in [5] formulated the following two problems (Open problems 2 and 3, page 243):

1. Show that no order $\prec \in \mathcal{L}_n$ is determined by $n-2$ (or less) binary subset comparisons.
2. Decide whether every $\prec \in \mathcal{L}_n$ which is determined by $n-1$ comparisons has the structure of the lexicographic ordering determined by

$$\{i_1, \ldots, i_j\} \prec i_{j+1}, \quad \text{for } j = 1, 2, \ldots, n-1.$$

We prove the first statement and show that the lexicographic ordering is only one of many orderings that can be determined by $n-1$ comparisons. We cannot characterise all of them but show that they are in one-to-one correspondence with the comparative probability orderings on the set consisting of $n-1$ elements determined by $n-1$ comparisons.

2 Preliminaries

Definition 1. *Let X be a finite set. A linear order $\prec$ on 2^X is called a comparative probability ordering on X if $\emptyset \prec A$ for every non-empty subset A of X, and $\prec$ satisfies de Finetti's axiom, namely*

$$A \prec B \iff A \cup C \prec B \cup C, \tag{1}$$

for all $A, B, C \in 2^X$ such that $(A \cup B) \cap C = \emptyset$.

For convenience, we will further suppose that $X = [n] = \{1, 2, \ldots, n\}$ and denote the set of all comparative probability orderings on $2^{[n]}$ as $\mathcal{P}_n$.

If we have a probability measure $\mathbf{p} = (p_1, \ldots, p_n)$ on X, where p_i is the probability of i, then we know the probability of every event A, by the rule $p(A) = \sum_{i \in A} p_i$. We may now define a relation $\preceq$ on 2^X by

$$A \preceq B \quad \text{if and only if} \quad p(A) \leq p(B).$$

If $p_i > 0$ for all i, and the probabilities of all events are different, then $\preceq$ is a comparative probability ordering on X, and we will denote it as $\prec_{\mathbf{p}}$. Any such ordering is called *(additively) representable*. The set of representable orderings is denoted $\mathcal{L}_n$. It is known [6] that $\mathcal{L}_n$ is strictly contained in $\mathcal{P}_n$ for all $n \geq 5$.

We will always assume here that a linear order $\prec$ on 2^X is a comparative probability ordering. As in [3, 4], it is often convenient to assume that

$$1 \prec 2 \prec \ldots \prec n, \tag{2}$$

which is equivalent to assuming that $p_1 < p_2 < \ldots < p_n$ when $\prec$ is a comparative probability ordering represented by the probability measure $\mathbf{p} = (p_1, \ldots, p_n)$. The set of all comparative probability orderings on $[n]$ that satisfy (2), will be denoted $\mathcal{L}_n^*$.

To every linear order $\prec \in \mathcal{L}_n^*$, there corresponds a *discrete cone* $C(\prec)$ in T^n, where $T = \{-1, 0, 1\}$ (as defined in [3]).

Definition 2. *A subset $C \subseteq T^n$ is said to be a discrete cone if the following properties hold:*

D1. $\{\mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_1, \ldots, \mathbf{e}_n - \mathbf{e}_{n-1}\} \subseteq C$, *where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$,* D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ *for every* $\mathbf{x} \in T^n$,

D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever* $\mathbf{x}, \mathbf{y} \in C$ *and* $\mathbf{x} + \mathbf{y} \in T^n$.

We note that in [3] Fishburn requires $\mathbf{0} \notin C$ because his orders are anti-reflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

For each subset $A \subseteq X$ we define the characteristic vector χ_A of this subset by

$$\chi_A(i) = \begin{cases} 1 & i \in A, \\ 0 & i \notin A, \end{cases}$$

$i = 1, 2, \ldots, n$. Given a comparative probability ordering $\prec$ on X, we define a characteristic vector $\chi(A, B) = \chi_B - \chi_A \in T^n$ for every possible comparison $A \prec B$. The set of all characteristic vectors $\chi(A, B)$, for $A, B \in 2^X$ such that $A \prec B$, is denoted as $C(\prec)$. The two axioms of comparative probability guarantee that $C(\prec)$ is a discrete cone (see [3, Lemma 2.1]).

Definition 3. *A comparative probability ordering $\prec$ satisfies the mth cancellation condition C_m if and only if there is no set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ of non-zero vectors in $C(\prec)$ for which there exist positive integers $a_1, \ldots, a_m$ such that*

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m = 0. \tag{3}$$

It is known [6, 3, 1] that a comparative probability ordering $\prec$ is representable if and only if all cancellation conditions for $C(\prec)$ are satisfied. The following condition is a reformulation of Axiom 3 in [5] in terms of discrete cones associated with $\prec$.

Lemma 1. *Let $\prec \in \mathcal{L}_n^*$ be a representable comparative probability ordering and $C(\prec)$ the corresponding discrete cone. Suppose $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq C(\prec)$ and suppose that for some positive rational numbers $a_1, \ldots, a_m$ and $\mathbf{x} \in T^n$*

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_m\mathbf{x}_m. \tag{4}$$

Then $\mathbf{x} \in C(\prec)$.

Proof. Let $a_i = p_i/q_i$, where p_i and q_i are positive integers. Then multiplying by the least common multiple of all the denominators, (4) can be written as

$$s\mathbf{x} = s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_m\mathbf{x}_m, \tag{5}$$

where $s_0, s_1, \ldots, s_m$ are integers. Suppose $\mathbf{x} \notin C(\prec)$. Then $-\mathbf{x} \in C(\prec)$ and (5) can be written as

$$s_0(-\mathbf{x}) + s_1\mathbf{x}_1 + s_2\mathbf{x}_2 + \cdots + s_m\mathbf{x}_m = \mathbf{0},$$

which contradicts the $(m+1)$th cancellation condition. $\square$

3 What does it mean that a set of comparisons determine the order?

This will be better explained algebraically, in terms of generators of cones.

Suppose that $C = C(\prec)$ is a discrete cone, with $\prec \in \mathcal{P}_n$ not necessarily representable. Then the only way we can deduce one comparison from several others is by means of transitivity. Use of transitivity corresponds to the addition of the corresponding characteristic vectors of the cone. Indeed, suppose that $A \prec B \prec C$. Then $\chi(A, B) + \chi(B, C) = \chi(A, C)$.

If $\prec$ is known to be representable, then there is an additional way to deduce a comparison from several others, using the cancellation conditions. This can be formulated in terms of multilists of comparisons (see, for example, [5], proof of Theorem 3.7). In terms of cones this tool is given in Lemma 1. It says that in representable cones we can deduce new comparisons by forming linear combinations of the characteristic vectors of known comparisons with positive coefficients.

We will refer to these as weak and strong generation respectively.

Let us define a restricted sum for vectors in a discrete cone C. Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u} + \mathbf{v} & \text{if } \mathbf{u} + \mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u} + \mathbf{v} \notin T^n. \end{cases}$$

Definition 4. *We say that the cone C is weakly generated by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if every nonzero vector $\mathbf{c} \in C$ can be expressed as a restricted sum of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, in which each generating vector can be used as many times as needed. We denote this by $C = <\mathbf{v}_1, \ldots, \mathbf{v}_k>_w$.*

We say that C is strongly generated by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if every nonzero vector $\mathbf{c} \in C$ can be deduced from $\mathbf{v}_1, \ldots, \mathbf{v}_k$ by forming restricted sums and applying Lemma 1.

A set of comparisons $A_1 \prec B_1, \ldots, A_\ell \prec B_\ell$ determine $\prec$ in $\mathcal{P}_n$ if the characteristic vectors $\chi(A_i, B_i)$ weakly generate $C(\prec)$. If $\prec$ is representable, the comparisons $A_1 \prec B_1, \ldots, A_\ell \prec B_\ell$ determine $\prec$ in $\mathcal{L}_n$ if the vectors $\chi(A_i, B_i)$ strongly generate $C(\prec)$.

Example 1. *Let us consider the ordering*

$$\emptyset \prec 1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 23 \prec 123.$$

and its respective cone $C = C(\prec)$. Let us choose $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (-1, 1, 0)$, $\mathbf{v}_3 = (-1, -1, 1)$, which correspond to comparisons

$$\emptyset \prec 1, \quad 1 \prec 2, \quad 12 \prec 3, \tag{6}$$

respectively. Then all other nonzero vectors of C can be expressed using $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. For example,

$$\begin{aligned}
\mathbf{v}_1 \oplus \mathbf{v}_2 &= (0,1,0),\\
\mathbf{v}_1 \oplus \mathbf{v}_3 &= (0,-1,1),\\
\mathbf{v}_2 \oplus \mathbf{v}_3 &= (-1,0,1),\\
\mathbf{v}_1 \oplus (\mathbf{v}_1 \oplus \mathbf{v}_2) &= (1,1,0),\\
(\mathbf{v}_1 \oplus (\mathbf{v}_1 \oplus \mathbf{v}_2)) \oplus \mathbf{v}_3 &= (0,0,1),
\end{aligned}$$

etc. Thus $C = <\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3>_w$. The ordering $\prec$ is representable, but we did not use linear combinations to generate C. Therefore the comparisons (6) determine $\prec$ not only in the class of representable orderings $\mathcal{L}_n$ but in the class of all comparative probability orderings $\mathcal{P}_n$.

Let us now give an example which shows that for representable cones weak and strong generation are different. We will construct a representable cone whose minimal set of weak generators will not be a minimal set of strong generators.

Example 2. *In the example constructed by Kraft, Pratt and Seidenberg ([6], page 415) we re-label $q = 1$, $r = 2$, $s = 3$, $p = 4$ and $t = 5$ to obtain a non-representable comparative probability ordering $\prec$ on* [5]*:*

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 4 \prec 14 \prec 23 \prec 5$$

$$\prec 123 \prec 24 \prec 34 \prec 15 \prec 124 \prec 25 \prec 134 \ldots$$

(where only the first 17 terms are shown). It does not satisfy the 4th cancellation condition since it contains the following comparisons:

$$13 \prec 4,\ 14 \prec 23,\ 34 \prec 15,\ 25 \prec 134,$$

whose corresponding characteristic vectors $\mathbf{u}_1 = (-1,0,-1,1,0)$, $\mathbf{u}_2 = (-1,1,1,-1,0)$, $\mathbf{u}_3 = (1,0,-1,-1,1)$, $\mathbf{u}_4 = (1,-1,1,1,-1)$ satisfy

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}.$$

The structure of the corresponding cone $C = C(\prec)$ is as follows: it includes all vectors of T^n lying in the half-space $S_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x}, \mathbf{b}) > 0\}$ and the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ lying on the hyperplane $H_{\mathbf{b}} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x}, \mathbf{b}) = 0\}$ with the normal vector

$$\mathbf{b} = \frac{1}{16}(1,2,3,4,6).$$

In this linear ordering 25 is the 16th subset and 134 is the 17th so $25 \prec 134$ is the central comparison of this ordering. This comparison can be reversed (see [7]), that is, we can replace $25 \prec 134$ with $134 \prec 25$ and still have a comparative probability ordering $\prec'$ with the cone $C' = (C \setminus \{\mathbf{u}_4\}) \cup \{-\mathbf{u}_4\}$. Moreover, this will be a representable comparative probability ordering since all cancellation conditions will be satisfied. Indeed, if we had a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subset C'$ such that

$$\mathbf{v}_1 + \mathbf{v}_2 + \ldots + \mathbf{v}_k = \mathbf{0},$$

then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, -\mathbf{u}_4\}$, which is impossible.

It is clear that any set of weak generators of C' must include all of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $-\mathbf{u}_4$. However

$$-\mathbf{u}_4 = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3$$

and hence $-\mathbf{u}_4$ can be excluded from the list of strong generators. It cannot be excluded from any list of weak generators since $\mathbf{u}_1 + \mathbf{u}_2 \notin T^n$, $\mathbf{u}_1 + \mathbf{u}_3 \notin T^n$, and $\mathbf{u}_2 + \mathbf{u}_3 \notin T^n$.

Definition 5. *Let C be a discrete cone. Define its weak rank, $\mathrm{rank}_w(C)$, to be the minimal number of vectors in C that weakly generate C. Define its strong rank, $\mathrm{rank}_s(C)$, to be the minimal number of vectors in C that strongly generate C.*

Obviously $\mathrm{rank}_s(C) \le \mathrm{rank}_w(C)$, and the previous example shows that they may be different.

Definition 6. *Let A and B be disjoint subsets of $[n]$. The pair (A, B) is said to be* critical *for $\prec$ if $A \prec B$ and for no $C \subseteq [n]$ is $A \prec C \prec B$.*

Now we can give an easy proof of the important Theorem 3.7 of [5].

Theorem 1 (Fishburn). *Let $\prec\ \in \mathcal{L}_n$ be a representable comparative probability ordering. Suppose that $A_1 \prec B_1$, ..., $A_\ell \prec B_\ell$ is the smallest set of subset comparisons that uniquely determines $\prec$ in $\mathcal{L}_n$. Then the pairs (A_1, B_1), ..., (A_ℓ, B_ℓ) are critical for $\prec$.*

Proof. The smallest set of subset comparisons corresponds to a minimal set of strong generators $G = \{\mathbf{g}_1, \ldots, \mathbf{g}_\ell\}$ of the respective cone $C = C(\prec)$. By the comment in the beginning of this section, to show that a pair (A_1, B_1) is critical, it is enough to show that $\mathbf{g}_1 = \mathbf{u} + \mathbf{v}$ for no two vectors $\mathbf{u}, \mathbf{v} \in C$. Let us assume the contrary. Then $m\mathbf{u} = \sum_{i=1}^{\ell} m_i \mathbf{g}_i$ and $k\mathbf{v} = \sum_{i=1}^{\ell} k_i \mathbf{g}_i$ for some positive integers m, m_i and k, k_j. Let $r = \mathrm{lcm}\,(m, k)$. Then, multiplying the two equations by their respective factors we obtain $r\mathbf{u} = \sum_{i=1}^{\ell} m_i' \mathbf{g}_i$ and $r\mathbf{v} = \sum_{i=1}^{\ell} k_i' \mathbf{g}_i$. Adding these we obtain

$$r\mathbf{g}_1 = \sum_{i=1}^{\ell} (m_i' + k_i')\mathbf{g}_i.$$

We now consider two cases. If $r > m'_1 + k'_1$, then $\mathbf{g}_1$ can be excluded from the set of generators G which was supposed to be minimal, contradiction. If $r \le m'_1 + k'_1$, then the ℓth cancellation condition is violated, contradiction again. This proves the theorem. $\square$

4 The Product of Two Orderings

Definition 7. *Suppose we have two comparative probability orderings $\prec_1 \in \mathcal{P}_k$ and $\prec_2 \in \mathcal{P}_m$. Let us define a new comparative probability ordering $\prec = \prec_1 \times \prec_2$ on $\mathcal{P}_{k+m}$ as follows. First we transfer the ordering $\prec_2$ from the set $[m]$ to the set $\{k+1, k+2, \dots, k+m\}$ in the obvious way. For any set $A = \{i_1, \dots, i_s\} \subseteq [m]$ we define its "shift" $\bar{A} = \{i_1+k, i_2+k, \dots, i_s+k\} \subseteq \{k+1, k+2, \dots, k+m\}$. We define $\bar{A} \prec_2 \bar{B}$ if and only if $A \prec_2 B$. It invites no confusion to call both the original order and the shifted one by the same name. Now, let $A, B \in [k+m]$. Then they can be uniquely represented as $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where $A_1, B_1 \in [k]$ and $A_2, B_2 \in \overline{[k]}$. Then $A \prec B$ if and only if $A_2 \prec_2 B_2$, or $A_2 = B_2$ and $A_1 \prec_1 B_1$.*

Definition 8. *The ordering $\prec \in \mathcal{L}_n$ is called* reducible *if it can be represented as a product of two other orderings.*

Example 3. *The lexicographic ordering $\prec \in \mathcal{L}_n$ defined in the introduction can be represented as*

$$\prec = (\dots(\prec_0 \times \prec_0) \times \prec_0) \times \dots) \times \prec_0,$$

where $\prec_0$ is the only ordering in $\mathcal{L}_1$, namely: $\emptyset \prec_0 \{1\}$.

Definition 9. *Let $\prec_1 \in \mathcal{P}_k$ and $\prec_2 \in \mathcal{P}_m$ be two comparative probability orderings with respective discrete cones $C_1 \subset T^k$ and $C_2 \subset T^m$. Define $C_1 \times C_2 \subset T^{k+m}$ to be the discrete cone of the product $\prec_1 \times \prec_2$.*

The cone $C_1 \times C_2$ consists of all vectors $(\mathbf{g}, \mathbf{0})$, where $\mathbf{g} \in C_1$ and all vectors $(\mathbf{g}, \mathbf{h})$, where $\mathbf{g} \in T^k$ and $\mathbf{h} \in C_2$.

Theorem 2. *$rank(C_1 \times C_2) = rank(C_1) + rank(C_2)$, where the rank can be either weak or strong.*

Proof. We will prove the theorem for the case of a weak rank. The proof for the case of a strong rank is similar.

Let $\{\mathbf{g}_1, \dots, \mathbf{g}_s\}$ and $\{\mathbf{h}_1, \dots, \mathbf{h}_t\}$ be minimal sets of weak generators for C_1 and C_2 respectively. Let G be the $s \times k$ matrix whose rows are the generators $\{\mathbf{g}_1, \dots, \mathbf{g}_s\}$ and H be the $t \times m$ matrix whose rows are the generators $\{\mathbf{h}_1, \dots, \mathbf{h}_t\}$. Let $1_{t\times k}$ be the $t \times k$ matrix whose all entries are 1, and $0_{s\times m}$ be an $s \times m$ zero matrix. Now we are going to check that the rows of the matrix

$$M = \begin{bmatrix} G & 0_{s\times m} \\ -1_{t\times k} & H \end{bmatrix}$$

generate $C_1 \times C_2$.

It is obvious that any row $(\mathbf{g}, \mathbf{0})$, where $\mathbf{0} \ne \mathbf{g} \in C_1$ can be obtained from the first s rows using the operation $\oplus$. In particular, the row $(\mathbf{1}, \mathbf{0})$ can be obtained, where $\mathbf{1}$ is the k-dimentional whose all entries are 1. Firstly, we will show that every vector $(-\mathbf{1}, \mathbf{h})$, where $\mathbf{0} \ne \mathbf{h} \in C_2$ can be generated. For this we need to show that if $(-\mathbf{1}, \mathbf{h}_1)$ and $(-\mathbf{1}, \mathbf{h}_2)$ can be generated and $\mathbf{h}_1 \oplus \mathbf{h}_2$ is defined, then also $(-\mathbf{1}, \mathbf{h}_1 \oplus \mathbf{h}_2)$ can be generated. To show this we note that

$$(-\mathbf{1}, \mathbf{h}_1 \oplus \mathbf{h}_2) = ((\mathbf{1}, \mathbf{0}) \oplus (-\mathbf{1}, \mathbf{h}_1)) \oplus (-\mathbf{1}, \mathbf{h}_2).$$

Suppose now that $\mathbf{c} = (\mathbf{g}, \mathbf{h}) \in T^{k+m}$, where $\mathbf{g} \in T^k$ and $\mathbf{0} \ne \mathbf{h} \in C_2$. Let us show that $\mathbf{c}$ can be generated using the rows of M as generators. It is not difficult to see that there exist vectors $\mathbf{g}_1, \mathbf{g}_2 \in C_1$ such that

$$\mathbf{g} = (-\mathbf{1} \oplus \mathbf{g}_1) \oplus \mathbf{g}_2.$$

Then $\mathbf{c}$ can be obtained as follows:

$$\mathbf{c} = (\mathbf{g}, \mathbf{h}) = ((-\mathbf{1}, \mathbf{h}) \oplus (\mathbf{g}_1, \mathbf{0})) \oplus (\mathbf{g}_2, \mathbf{0}).$$

It is clear that the set of generators given as rows of M is minimal. $\square$

Example 4. *Let us consider the ordering*

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 23 \prec 123.$$

and its respective cone $C = C(\prec)$. This cone has the following set of generators (both weak and strong):

$$\begin{aligned} \mathbf{h}_1 &= (-1, 1, 0), \\ \mathbf{h}_2 &= (0, -1, 1), \\ \mathbf{h}_3 &= (1, 1, -1). \end{aligned}$$

Let $\prec_0$ be the ordering $\emptyset \prec_0 1$ of $\mathcal{L}_1$. Then apropos of the proof of Theorem 2 the rows $\mathbf{u}_1, \dots, \mathbf{u}_4$ of the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

generate the cone for the ordering $\prec_0 \times \prec$.

We conclude this section with the following

Theorem 3. *The product of two representable comparative probability orderings $\prec_1 \in \mathcal{P}_k$ and $\prec_2 \in \mathcal{P}_m$ is a representable comparative probability ordering of $\mathcal{L}_{k+m}$.*

Proof. Let $\mathbf{p} = (p_1, \ldots, p_k)$ and $\mathbf{q} = (q_1, \ldots, q_m)$ be the probability measures that represent $\prec_1$ and $\prec_2$, respectively. Since both orderings are linear, the following number is non-zero

$$\varepsilon = \min_{I,J} \left\{ \left| \sum_{i\in I} q_i - \sum_{j\in J} q_j \right| \right\}, \ I, J \subseteq [m], \ I \cap J = \emptyset.$$

It is easy to check that the measure given by

$$\frac{1}{\varepsilon + 1} (\varepsilon p_1, \varepsilon p_2, \ldots, \varepsilon p_k, q_1, q_2, \ldots, q_m)$$

defines the ordering $\prec_1 \times \prec_2$. □

5 Main Results

We recall a few basic facts about hyperplane arrangements in $\mathbb{R}^n$ (see [8] for more information about them). Any hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{n}, \mathbf{x}) = 0\}$, where $\mathbf{n}$ is a non-zero vector, we will call *linear.* The vector $\mathbf{n}$ is called the *normal vector* of H. It is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. If $a \neq 0$, any hyperplane $J = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{n}, \mathbf{x}) = a\}$ will be called an *affine* hyperplane. An affine hyperplane is a translate of the linear hyperplane with the same normal vector.

A *hyperplane arrangement* $\mathcal{A}$ is any finite set of hyperplanes. A *region* of an arrangement $\mathcal{A}$ is a connected component of the complement of the union U of the hyperplanes of $\mathcal{A}$, i.e., the set

$$U = \mathbb{R}^n \setminus \bigcup_{H\in\mathcal{A}} H.$$

Any region of an arrangement is an open set.

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^n$ and J be a hyperplane in $\mathbb{R}^n$. Then the set

$$\mathcal{A}^J = \{H \cap J \mid H \in \mathcal{A}\}$$

is called the *induced arrangement of hyperplanes* in J.

Let $A, B \subseteq [n]$ be disjoint subsets, of which at least one is non-empty. We put in correspondence with this pair the hyperplane $H(A, B)$ in $\mathbb{R}^n$ given by the equation

$$\sum_{a\in A} x_a - \sum_{b\in B} x_b = 0.$$

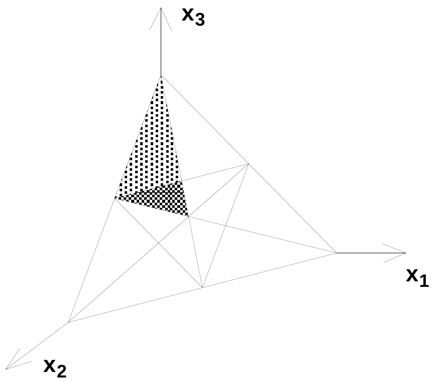
These hyperplanes have normal vectors in the set $\{-1, 0, 1\}^n \setminus \mathbf{0}$, where $\mathbf{0}$ is the n-dimensional zero vector. Let us denote the corresponding hyperplane arrangement by $\mathcal{A}$. Let J be the hyperplane defined by the equation $x_1 + x_2 + \ldots + x_n = 1$ and $\mathcal{A}^J$ be the induced hyperplane arrangement. We are interested in the regions of $\mathcal{A}^J$ which lie in the positive orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$, given by $x_i \geq 0$, $i = 1, 2, \ldots, n$. These regions altogether form a simplex S in J.

Every point $\mathbf{p} = (p_1, \ldots, p_n) \in S$ defines a representable comparative probability ordering $\prec_\mathbf{p}$ from $\mathcal{L}_n$, the one which obtains when we allocate measure p_i to i for $i = 1, 2, \ldots, n$. If $\mathbf{p}$ and $\mathbf{q}$ are two points from S, then the orderings $\prec_\mathbf{p}$ and $\prec_\mathbf{q}$ will coincide if and only if $\mathbf{p}$ and $\mathbf{q}$ are in the same region of the hyperplane arrangement $\mathcal{A}^J$. This immediately follows from the fact that the order $A \prec B$ changes to $B \prec A$ (or the other way around) if and only if we cross the hyperplane $H(A, B)$. Thus every comparative probability ordering from $\mathcal{L}_n$ is so represented by one of the regions. Every such region is a convex polytope.

Let $A, B \subseteq [n]$ be two non-empty disjoint subsets. Then we can have both $A \prec B$ and $B \prec A$, so the comparison of A and B gives us certain information about the ordering. However, if $A = \emptyset$ and B is non-empty, then $A \prec B$ and such comparison gives no information about the ordering (as it is axiomatic). The latter comparisons correspond with the hyperplanes $x_i = 0$, for $i = 1, 2, \ldots, n$.

Let P be the polytope representing $\prec_\mathbf{p}$. A face of the polytope will be called *significant* if it is not contained in any of the hyperplanes $x_i = 0$. It is now clear that the minimal number of subset comparisons needed to define $\prec_\mathbf{p}$ is the number of significant faces of the polytope representing $\prec_\mathbf{p}$. We illustrate this in the following example.

Example 5. *The 12 regions on the figure below represent all 12 comparative probability orderings on* [3].



The two shaded triangular regions correspond to the two orderings

$$1 \prec 2 \prec 12 \prec 3 \prec 13 \prec 23 \prec 123,$$

$$1 \prec 2 \prec 3 \prec 12 \prec 13 \prec 23 \prec 123,$$

which satisfy $1 \prec 2 \prec 3$*, with the lighter one corresponding to the first (lexicographic) ordering. However one of the boundaries for the lexicographic ordering is* $x_1 = 0$ *and it is determined by two comparisons, while the other order needs three comparisons since all faces of the corresponding region are significant.*

A simple but important statement is contained in the following

Lemma 2. *Let* P *be the polytope representing the ordering* $\prec_{\mathbf{p}}$*, where* $\mathbf{p}$ *is a probability measure. Then* P *can have at most one insignificant face.*

Proof. Indeed, if $p_j = \min_{i=1}^{n} p_i$, then one of its faces is contained in the hyperplane $x_j = 0$ and this is the only insignificant face of P. Suppose that another face of P is contained in $x_k = 0$. Since P lies in the hyperplane $x_k - x_j \geq 0$, we will have $x_k = x_j = 0$ on this face. Since this face is also in J, it cannot have a nonzero $(n-2)$-dimensional volume. However every face of an $(n-1)$-dimensional polytope must have a nonzero $(n-2)$-dimensional volume. □

The following theorem answers Fishburn's first question.

Theorem 4. *Let* $\prec$ *be a representable comparative probability ordering. Then it is determined by at least* $n-1$ *binary subset comparisons* $A \prec B$*, where* A, B *are disjoint non-empty subsets of* $[n]$*. If all faces of the polytope* P *representing* $\prec$ *are significant, then* $\prec$ *is determined by at least* n *binary subset comparisons.*

Proof. Consider the hyperplane J defined by the equation $x_1 + x_2 + \ldots + x_n = 1$, and the simplex $S = J \cap \mathbb{R}_+^n$ and the induced partitioning of S into regions by the hyperplanes of the arrangement $\mathcal{A}^J$. Let $\mathbf{p}$ be a probability measure that corresponds to a representable ordering $\prec \in \mathcal{L}_n$. The region P to which $\mathbf{p}$ belongs is an open convex polytope. Since it is open, it has a nonzero $(n-1)$-dimensional volume in J and, therefore must have at least n vertices. (Indeed if P only had the vertices $A_1, \ldots, A_k$, where $k < n$, then the $k-1$ vectors $\overrightarrow{A_1A_2}, \ldots, \overrightarrow{A_1A_k}$ are linearly dependent and the polytope P has zero volume).

Now, having at least n vertices, P must have at least n faces. Indeed, let $H_1, \ldots, H_k$ be the hyperplanes that contain faces of P. Let A be any vertex of P. Then the collection of hyperplanes to which A belongs have a unique point of intersection, which is A. In an $n-1$-dimensional hyperplane J one needs at least $n-1$ hyperplanes to intersect in a point. Hence P has at least $n-1$ faces containing A. Since P is bounded, there must be at least one other face.

If all faces are significant, each face corresponds to a non-trivial subset comparison, hence we need at least n subset comparisons to determine $\prec$. Otherwise by Lemma 2 there is a single insignificant face. In this case $n-1$ subset comparisons are needed to determine $\prec$. □

For orderings in $\mathcal{L}_n^*$, i.e. for those for which (2) is satisfied, we have the following

Corollary 1. *Let* $\prec$ *be a representable comparative probability ordering in* $\mathcal{L}_n^*$ *which is determined by exactly* $n-1$ *binary subset comparisons. Then* $x_1 = 0$ *contains one of the faces of the region which corresponds to* $\prec$*.*

This can be expressed in terms of discrete cones as follows:

Corollary 2. *Let* $\prec$ *be a representable comparative probability ordering in* $\mathcal{L}_n^*$ *which is determined by exactly* $n-1$ *binary subset comparisons. Then the vector* $\mathbf{g}_1 = (1, 0, \ldots 0)$ *is present in any set of generators of the corresponding discrete cone* $C(\prec)$*.*

Now we can give a structural characterisation of the comparative probability orderings from $\mathcal{L}_n^*$ whose corresponding polytope has one of its faces contained in $x_1 = 0$.

Theorem 5. *Let* $\prec$ *be a comparative probability orderings from* $\mathcal{L}_n^*$ *whose corresponding polytope* P *has one of its faces contained in* $x_1 = 0$*. Then* $\prec \,=\prec_0 \times \prec'$*, where* $\prec_0$ *is the only ordering in* $\mathcal{L}_1$ *and* $\prec'$ *is a comparative probability ordering from* $\mathcal{L}_{n-1}^*$*.*

Proof. Let S be the minimal set of comparisons that define $\prec$. By Theorem 1 all comparisons in S are critical. Let $A \prec B$ be any one comparison from S, where $A, B \subseteq [n]$ are disjoint and non-empty. Let us consider the hyperplane

$$\sum_{a \in A} x_a = \sum_{b \in B} x_b.$$

which correspond to this comparison. Since $\mathbf{p}$ does not lie on this hyperplane, assume without loss of generality that

$$\sum_{a \in A} p_a < \sum_{b \in B} p_b.$$

Let us show that we also have

$$\sum_{a \in A \cup \{1\}} p_a < \sum_{b \in B \setminus \{1\}} p_b. \tag{7}$$

First, imagine that $1 \in B$. Since $A \prec B$ is critical, we have $B \setminus \{1\} \prec A$, hence

$$\sum_{b \in B \setminus \{1\}} p_b < \sum_{a \in A} p_a < \sum_{b \in B} p_b.$$

Then

$$\sum_{b\in B\setminus\{1\}} x_b < \sum_{a\in A} x_a < \sum_{b\in B} x_b. \tag{8}$$

for every interior point $\mathbf{x}$ of this region and

$$\sum_{b\in B\setminus\{1\}} x_b \le \sum_{a\in A} x_a \le \sum_{b\in B} x_b. \tag{9}$$

on the faces of P. In particular, (9) must be true on the face $x_1 = 0$, which is a contradiction since no internal point of that face is on the hyperplane $\sum_{b\in B\setminus\{1\}} x_b = \sum_{a\in A} x_a$. Similarly, imagine that $1 \notin B$. Then similar considerations lead to

$$\sum_{a\in A} p_a < \sum_{b\in B} p_b < \sum_{a\in A\cup\{1\}} p_a,$$

and a contradiction can be obtained in a similar way.

It follows that $1 \in A$, since assuming the contrary will make the comparison $A \prec B$ non-critical, which is a contradiction.

Now, consider $C = C(\prec)$. We know that this cone has n strong generators, $\{\mathbf{g}_1, \dots, \mathbf{g}_n\}$, with $\mathbf{g}_1 = (1, 0, \dots 0)$. The other strong generators $\mathbf{g}_2, \dots, \mathbf{g}_n$ correspond to characteristic vectors of critical pairs $A_i \prec B_i$ from S with non-empty A_i and B_i. We proved that $1 \in A_i$ for all $i = 2, \dots, n$, hence the corresponding generators will be $\mathbf{g}_i = (-1, \mathbf{g}'_i)$ for all $2 \le i \le n$, where $\mathbf{g}'_i \in T^{n-1}$. Thus the rows of the following matrix are the strong generators of C.

$$G = \begin{bmatrix} 1 & 0 \\ -1 & \mathbf{g}'_2 \\ \vdots & \vdots \\ -1 & \mathbf{g}'_n \end{bmatrix}$$

Clearly $\mathbf{g}'_2, \dots \mathbf{g}'_n$ generate a cone $C' = C(\prec')$ for some $\prec' \in \mathcal{L}_{n-1}$. As $\mathrm{rank}(C') \le n-1$, the ordering $\prec'$ is determined by no more than $n-1$ comparisons. The form of G demonstrates that $C = C(\prec_0) \times C'$, and so $\prec = \prec_0 \times \prec'$. $\square$

Now we can give a characterisation of the comparative probability ordering from $\mathcal{L}_n^*$ that can be determined by $n-1$ binary comparisons, answering Fishburn's second question.

Theorem 6. *Let $\prec$ be a comparative probability ordering from $\mathcal{L}_n^*$ that can be determined by $n-1$ binary comparisons. Then it is reducible and $\prec = \prec_0 \times \prec'$, where $\prec_0$ is the only ordering in $\mathcal{L}_1$ and $\prec'$ is a comparative probability ordering from $\mathcal{L}_{n-1}^*$ that can be determined by no more than $n-1$ comparisons. Conversely, whenever we have $\prec' \in \mathcal{L}_{n-1}^*$ determined by no more than $n-1$ comparisons, then $\prec = \prec_0 \times \prec'$ is a comparitive probability ordering in $\mathcal{L}_n^*$ that can be determined by $n-1$ binary comparisons, unless $\prec'$ requires exactly $n-1$ comparisons and is reducible to $\prec' = \prec_0 \times \prec''$.*

Proof. Let $\prec \in \mathcal{L}_n^*$ be any order determined by a set of $n-1$ binary comparisons. Let $\mathbf{p}$ be a probability measure that determines $\prec$, and P the corresponding polytope in J. Then Corollary 1 implies that $x_1 = 0$ is a face of P. By Theorem 5 we know that $\prec = \prec_0 \times \prec'$, where $\prec_0$ is the only ordering in $\mathcal{L}_1$ and $\prec'$ is a comparative probability ordering from $\mathcal{L}_{n-1}^*$. Let $C' = C(\prec')$. Then $rank(C') = n-1$ and $\prec'$ is determined by $n-1$ or $n-2$ comparisons.

For the converse, if $\prec' \in \mathcal{L}_{n-1}$ is determined by at most $n-1$ comparisons, we will have:

Case 1 $\prec'$ is determined by $n-1$ comparisons and $\prec' \neq \prec_0 \times \prec''$. Then if H is a matrix with $n-1$ rows that are strong generators of $C(\prec')$,

$$G = \begin{bmatrix} 1 & 0 \\ -1 & H \end{bmatrix}$$

generates the cone $C = C(\prec_0 \times \prec')$. However, the first row is $\mathbf{g}_1 = (1, 0, \dots 0)$ and so $\prec = \prec_0 \times \prec'$ is determined by $n-1$ binary comparisons. (We do not have to add a row $(1, 0, \dots, 0)$ to H because the row $(-1, 1, 0, \dots, 0)$ is strongly implied from other rows of G.)

Case 2 $\prec'$ is determined by $n-2$ comparisons. Then, as we proved in Theorem 8 the cone $C(\prec')$ has $n-1$ generator $\mathbf{h}_1, \dots, \mathbf{h}_{n-1}$ with $\mathbf{h}_1 = (1, 0, \dots, 0)$. Then the rows of

$$G = \begin{bmatrix} 1 & 0 \\ -1 & H \end{bmatrix}$$

are generators of C and since its first row is $\mathbf{g}_1 = (1, 0, \dots 0)$, the corresponding ordering $\prec$ is determined by $n-1$ comparisons. $\square$

Theorem 7. *There are exactly* 2 *orders in $\mathcal{L}_4^*$ that can be determined by* 3 *binary comparisons, and exactly* 11 *orders in $\mathcal{L}_5^*$ that can be determined by* 4 *comparisons.*

Proof. There are 2 members of $\mathcal{L}_3^*$. The lexicographic order $\prec_1$ in $\mathcal{L}_3^*$ is determined by the two comparisons $1 \prec_1 2$ and $12 \prec_1 3$, and hence $\prec = \prec_0 \times \prec_1 \in \mathcal{L}_4$ is determined by $1 \prec 2$, $12 \prec 3$ and $123 \prec 4$. The other order $\prec_2$ of $\mathcal{L}_3$ is determined by the three comparisons $1 \prec_2 2$, $2 \prec_2 3$ and $3 \prec_2 12$, and hence $\prec = \prec_0 \times \prec_2 \in$

$\mathcal{L}_4^*$ is determined by $12 \prec 3$, $13 \prec 4$ and $14 \prec 23$. The 14 members of $\mathcal{L}_4^*$ are listed in [5]. Here we list the minimal set of determining binary comparisons for each.

1. $2 \prec 3$, $3 \prec 4$, $4 \prec 12$, $14 \prec 23$,
2. $1 \prec 2$, $2 \prec 3$, $4 \prec 12$, $23 \prec 14$,
3. $3 \prec 12$, $12 \prec 4$, $4 \prec 13$, $14 \prec 23$,
4. $1 \prec 2$, $3 \prec 12$, $12 \prec 4$, $4 \prec 13$, $23 \prec 14$,
5. $12 \prec 3$, $3 \prec 4$, $4 \prec 13$, $14 \prec 23$,
6. $1 \prec 2$, $12 \prec 3$, $4 \prec 13$, $23 \prec 14$,
7. $2 \prec 3$, $3 \prec 12$, $13 \prec 4$, $14 \prec 23$,
8. $2 \prec 3$, $3 \prec 12$, $13 \prec 4$, $4 \prec 23$, $23 \prec 14$,
9. $12 \prec 3$, $13 \prec 4$, $14 \prec 23$,
10. $12 \prec 3$, $13 \prec 4$, $4 \prec 23$, $23 \prec 14$,
11. $1 \prec 2$, $2 \prec 3$, $3 \prec 12$, $23 \prec 4$, $4 \prec 123$,
12. $1 \prec 2$, $2 \prec 3$, $3 \prec 12$, $123 \prec 4$,
13. $1 \prec 2$, $12 \prec 3$, $23 \prec 4$, $4 \prec 123$,
14. $1 \prec 2$, $12 \prec 3$, $123 \prec 4$.

In this list every order determined by 4 comparisons is irreducible, and therefore all 11 orders determined by ≤ 4 binary comparisons can be extended to orders in $\mathcal{L}_5^*$ determined by exactly 4 binary comparisons.

$\square$

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