Ordinal Subjective Foundations for Finite-domain Probability Agreement

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Abstract

Normative study of probability-agreeing orderings of propositions, much of it rooted in a false but evocative conjecture of Bruno de Finetti, has typically sought to abstract credal rationality claims familiarly made for numerical probabilities. It is now known that some probability-disagreeing orderings, *e.g.* possibilistic order, syntactically restate probability-agreeing orderings, and so share in any ordinal probabilistic 'rationality.' This paper explores what remains normatively distinctive about subjective probability agreement. A multiset partial ordering, characteristic of all transitive elementary orderings, helps provide succinct, apprehensible necessary and sufficient ordinal conditions for probability agreement.

Keywords. Qualitative probability, transitivity, de Finetti's conjecture, Scott's theorem.

1 Introduction

Bruno de Finetti (1935) developed a semantics and normative motivation for numerical subjective probability based upon orderings of propositions according to personal judgments about their relative credibility. Prominent in his account was a property often called *quasi-additivity*,

> For propositions A, B, and C, where A is exclusive of C and B is exclusive of C: A is no less credible than B just when A ∨ C is no less credible that B ∨ C.

Other properties used by de Finetti were transitivity, and what will be called here *boundedness*, that a tautology is more credible than an uncertain proposition and a contradiction is less credible than an uncertain proposition. De Finetti also assumed that credibility orderings were complete and definite, *i.e.* for any propositions A and B, $A > B$, $B > A$, or else $A = B$.

From these assumptions, de Finetti derived a specific numerical probability for an initial set of propositions, provided that unboundedly many auxiliary propositions, judged to be equally likely, were available to compare with the original domain. Later authors suggested that a suitable auxiliary domain might be achieved by imaginary coin tosses (Savage, 1972), or by the outright assumption of a uniform-probability real-valued random variable (De Groot, 1970).

De Finetti, however, subscribed to a rigorously subjective, personalist probabilism. He doubted the cogency of "objective" probability notions (1949). He would have been reluctant to introduce into his argument any figments like those suggested by the later authors. Except to insist on their subjectivity, De Finetti had been vague about the source of his auxiliary propositions.

Eventually, de Finetti put aside his ordinal-based project in favor of an influential gambling semantics for numerical subjective probability. Matters might have rested there but for a challenge pointedly posed by de Finetti's friend, George Polya (1949), who held that numbers should play little role in belief modeling.

In reply to Polya, de Finetti (1949) revived his ordinal studies, attempting to motivate a non-numerical subjective qualitative probability. The reply paper's propositional domain is finite, the 2^n distinct disjunctive propositions that can be composed from n mutually exclusive atomic propositions. The auxiliary domain of earlier ordinal work, whose only intended purpose had been to extend order into specific numbers, was absent in 1949.

De Finetti treated ordinal assertions as simultaneous linear inequality constraints. For example, the assertion $a \vee b > c$ becomes the constraint $p(a) + p(b) > p(c)$. Such linear systems had earlier been the vehicle by which de Finetti had developed his gambling semantics.

De Finetti conjectured that quasi-additivity, transitivity, and boundedness sufficed for a finite complete and definite ordering to be probability agreeing. Probability agreement occurs when there exists some probability distribution p() on the propositions in an ordering, such that $A \geq B$ in the ordering just when $p(A) \geq p(B)$.

De Finetti's conjecture is false. Kraft, Pratt, and Seidenberg (1959) constructed a quasi-additive, transitive, bounded, and definite ordering of the 32 propositions built on five atoms which included the assertions:

$a \vee c \vee d > b \vee e \qquad a \vee e > c \vee d$

$b \vee c > a \vee d \qquad d > a \vee c$

There is no agreeing probability. Summing the inequalities above yields the assertion that the quantity $2p(a)+p(b)+2p(c)+2p(d)+p(e)$ is strictly greater than itself, a contradiction among the constraints.

Kraft, *et al.* also showed that a finite variation upon de Finetti's auxiliary domain method, along with the conjecture's assumptions, sufficed for demonstrating the existence of an agreeing probability. The authors framed their argument to avoid comparisons between the original and the auxiliary events, and between conjunctions involving different original events (*e.g.* "heads-and-*A*" might be compared to "tails-and-*A*" but not to "heads-and-*B*").

These is reason to doubt whether this repair strategy, or any strategy featuring an auxiliary domain, achieved de Finetti's goals in 1949 for a transparently subjective foundation of qualitative probability. Recourse to coins, suggested by the Kraft group, would fall especially short of de Finetti's plausible goals.

Moreover, de Finetti had been aware that introducing an auxiliary domain could rescue his conjecture if the need arose (1949). He did not pursue the possibility. It seems likely that part of the beauty of the conjecture for de Finetti was the absence of any auxiliary domain congruent to the outcomes of some gambling apparatus, so avoiding that blemish upon subjectivist motivations which Cox (1946) criticized as "toolmarks" betraying an origin in frequentist intuition.

In any case, it follows immediately from Kraft, Pratt, and Seidenberg's results that:

Scott's Theorem (1964). Any finite system of simultaneous linear inequalities on a bounded domain, all of whose non-zero coefficients are one (as ordered disjunctions engender), has a solution ***unless*** there is some finite combination of constraints of the same weak sense where at least one inequality is strict, such that each of the atoms appears the same number of times on each side of the inequalities in the combination.

Scott obtained this 'uninterpreted' version of the key sufficient condition for probability agreement by distinct means, deferring to Kraft, Pratt, and Seidenberg's priority in the essence. Scott's theorem is described as 'uninterpreted' in that he declined to provide any normative argument to explain why one would ever want to count the number of times each atom appeared on each side of a system of inequalities. Although Scott did not provide such an argument, there nevertheless are suitable arguments he might have made, as will be discussed in the next section.

In the decades since then, many other authors have built on Kraft, *et al.* and Scott, seeking other sufficient conditions (often relaxing necessity) for probability agreement, along with answers to related questions, such as sufficient conditions for agreement with exactly one density. Fishburn (1996) includes a survey of, and pointers into, the breadth of this work.

What is sought in the present paper are necessary and sufficient conditions for finite probability agreement, of both qualitative and unambiguously subjective character, whose interpretation in the context of belief modeling should motivate what an additive measure might have to do with the relationships among ordered beliefs.

2 The Task Perspective

In fact, de Finetti and many of those who came after him clearly wished for something more. There was hope that there might be necessary and sufficient conditions for probability agreement that were also necessary for the achievement of rationality, however that estate might be defined.

It is now known that there are no such ordinal conditions. The propositional orderings of a linear possibility syntactically restate the ordering assertions of some probability densities which agree with the possibility on the order of the atoms (Snow, 2001). Possibility orders disjunctions according to the maximum value among their disjoined atoms; $\varnothing$ has a value of zero. In a linear possibility, all atomic values are distinct. Except for the Boolean distribution (exactly one non-zero atomic value), possibility is not probability agreeing.

The probability density whose atomic values' numerators are successive powers of two (for instance, with 5 uncertain atoms, values of 1/31, 2/31. 4/31, 8/31, and 16/31) relates to the atomically agreeing (and so linear) possibility by the rules

$\text{poss}(A) \geq \text{poss}(B) \Rightarrow \text{prob}(A\neg B) \geq \text{prob}(B)$

$\text{prob}(A) \geq \text{prob}(B) \Rightarrow \text{poss}(A\neg B) \geq \text{poss}(B\neg A)$

The ordinal assertions of either calculus can thus be mechanically translated into ordinal assertions of the other. Any ordinal thought expressible in one calculus finds its corresponding expression in the other. If either one is reasonable, then the other is, too, since both can express the same underlying ordinal thoughts, including each other's, with equal fidelity.

It also follows that the bald direction to order propositions according to their 'credibility' is too vague to be of much use in discovering specifically probabilistic ordinal principles. More helpful would be to assign some more specific task whose achievement is informed by beliefs and depends on probabilistic expression of those beliefs.

One would not claim that performance of any specific task is the sum and substance of rational belief tenure, but one might hope that it would be easier to judge whether or not the assigned task has been done well than to judge whether disputable notions of rationality have been served. De Finetti's (or others') gambling semantics would be an example of assigning a specific task in which good performance calls for probabilistic resemblance.

Consider the following task specification. The subject is presented with some number s of pairs of propositions. The assignment is to select one member of each pair so that as many of the s selected propositions as possible will be found true if and when all uncertainty is resolved, that is, some one atom is found true. If the subject judges the propositions in any pair to be equally suitable for achieving that objective, then either member may be selected, however the subject pleases. The subject is to 'show his or her work' so that we may distinguish between selections based on ties and otherwise.

Suppose that at least one non-tying selection has been made, and that the subject assents to the interpretation that he or she expects that more of the s selected propositions will be found true than of the s propositions left out. In the case of ties, if different choices were made, then the subject would see no advantage nor disadvantage compared to the selections actually made. If the subject wishes to revise his or her selections in order to attain assent, then that is fine.

While many solutions to the task are equally defensible, others may contain patterns of choice that would be anomalous if they occurred. Suppose, for example, there were a quasi-additivity violation: $a \vee b = a \vee c$ was asserted, $a \vee b$ being selected, and $c > b$ was asserted and c selected. This selection pattern is self-defeating with respect to the assigned goal.

If neither b nor c were found true, then these choices make no difference to the number of correct selected propositions. Otherwise, if choosing c over b contributes to the achievement of the goal, then choosing $a \vee b$ instead of $a \vee c$ detracts from it, and *vice versa*. The subject expects that he or she could have done the task better. How the subject would resolve the difficulty is of no concern. What rubs is that there is some room for improvement.

The difficulty is not that the simultaneous assertion of $a \vee b = a \vee c$ and $c > b$ violates some canon of rational belief. Possibilists might so assert, and possibility ordering is rational, even by probabilists' foundational standards. The difficulty is that the subject's beliefs have not successfully informed the achievement of the assigned task, a task involving a *bona fide* aspect of belief tenure, albeit not the only aspect.

The relationship to Scott's uninterpreted sufficiency criterion is clear. If every atom occurs just as many times among the s selected propositions as among the s rejected ones, then the same number of rejected propositions will necessarily be found true as accepted propositions, if and when the uncertainty is resolved. How many propositions will be found true will depend on which atom is found true, but the number among the accepted and the rejected will be the same, whichever atom is found true.

By the earlier supposition, the subject assented otherwise. The import of the situation is the same as for the hypothetical quasi-additivity violation, as it must be, since that violation consisted of both accepting and rejecting one instance apiece of a, b, and c. No canon of rational belief has been violated, but a *bona fide* aspect of belief tenure has been poorly managed.

For those who prefer a gambling story, suppose the task were posed as s opportunities to choose to be paid \$1 if some proposition is found true, or else to be paid \$1 if some other proposition is found true. The goal is to construct the highest-paying portfolio of s propositions from the pairs. The Scott-style anomaly would consist in judging that the accepted portfolio had strictly better prospects than the rejected one, while both portfolios will in fact result in identical pay-offs.

That is a less dramatic predicament than the usual Dutch book. Nevertheless, it is a predicament all the same, and yields an especially simple gambling basis for ordinal subjective probability.

With or without gambling, the arguments of this section share a theme with many other normative arguments for probability. Conflict arises between some goal for belief-based assertions collectively, opposed by the logical impossibility of the goal being realized without probabilistic conformity. The schema involves what might be described as testing the quality of belief against circumstances outside the mind of the believer.

Such testing is legitimate, and important for realistic application. Moreover, some role for a task more specific than to "order propositions according to judged credibility" is inevitable. However, just as de Finetti's conjecture lacked the nettlesome auxiliary domain, so also did it rely solely on assumptions about plausibly felt relationships among beliefs, rather than performance failures. A repair fully in the spirit of the original conjecture should do likewise.

An offer of repair will be made which focuses on the believer's reasoning when undertaking a task like that of this section, but does not turn on the assessment of the result that the believer produces. First, though, a helpful mathematical fact will be introduced.

3 An Aspect of Orderings

All transitive orderings of objects impose a partial order on the multisets of those objects favored and disfavored by ordering assertions. A multiset, also known as a bag, allows an object to appear more than once as an element, but elements are not ordered within the bags.

Two bags are equal just when they contain the same elements, each present the same number of times. The size of a bag is the number of elements it contains. It is straightforward to "type cast" a bag into an associated set of its distinct elements, or into a list which does impose some internal order on the contents of the bag.

Definition. For any finite transitively ordered domain of objects D, the *object-matching partial order* asserts, for same-size bags A and B of objects in D, that $A > B$ just when there is a bijection $f()$ from A to B in which for each element a in A, $a \geq f(a)$ in the ordering of D, and for some element the ordering is strict, and asserts that $A = B$ just when there is a bijection $f()$ from A to B where for each a in A, $a = f(a)$ in the ordering of D.

Proposition. For bags A and B of the above definition, at most one of $A = B$, $A > B$, or $B > A$ holds for definitely ordered objects.

A sketch proof of the Proposition appears in the Appendix. It is straightforward that the object-matching partial order is transitive, and that there is a binary concatenation operation U & V which produces the bag containing the elements of U and V, for which $A \geq B \Leftrightarrow A \,\&\, C \geq B \,\&\, C$.

4 Extending the Proposition-bag Partial Order

Throughout this section, whenever any ordering of propositions is discussed, the ordering will be complete, definite, bounded, and transitive. The domain will be finite.

When implementing the task of ordering propositions according to their prospects for being found true, the believer visibly resorts to compensation, or "trade offs." That is, in comparing $A \vee B$ with $C \vee D$ ($AB = CD = \varnothing$), one might assert that $A \vee B > C \vee D$ when $A > C$ but $B < D$, even when $A < C \vee D$.

The believer evidently reasons, or would testify if asked, that the advantage of A compared to C overcomes the advantage of D compared to B, leading to the conclusion that $A \vee B > C \vee D$. Or, in other words, if $D > B$ is a "closer call" than $A > C$, then this leads to $A \vee B$ having a better overall prospect of being found true than $C \vee D$. Analogous considerations would be salient when making subjective qualitative estimates of uncontroversially additive quantities like distances, weights, or the proportions of species in a mixture. To be formal:

Definition. A *contrast* is an ordered pair of propositions.

Assumption 1. There is a transitive ordering among the contrasts defined on the propositions of the domain such that for $AB = CD = \varnothing$, $A \vee B \geq C \vee D$ just when $(A, C) \geq (D, B)$.

Although we do not seek another testing-based motivation, contrasts do have interpretations "outside the mind." For example, (A, B) can be interpreted as the joint prospect of being paid \$1 if A is found true, and being liable to pay \$1 if B is found true (alternatively, being paid \$1 unless B is found true, if one wishes to avoid losses). This, combined with the interpretation of propositions as \$1 prospects, leads to gentle anomalies like those of an earlier section if the assumed relationship between contrasts and propositions does not hold.

Assumption 1 implies quasi-additivity. If $AC = BC = \varnothing$, and $A \vee C \geq B \vee C$, then we have $(A, B) \geq (C, C)$. Since $C = C$, $(C, C) = (\varnothing, \varnothing)$, so by transitivity, $(A, B) \geq (\varnothing, \varnothing)$, or $A \geq B$. The steps are reversible. Assumption 1 is also strictly stronger than quasi-additivity, since it excludes the Kraft, Pratt, and Seidenberg counterexample, which quasi-additivity does not:

$$a \vee e > c \vee d \quad \Rightarrow \quad \mathbf{(a, \varnothing)} > (c \vee d, e)$$

$$a \vee c \vee d > b \vee e \quad \Rightarrow \quad (c \vee d, e) > (b, a)$$

$$b \vee c > a \vee d \quad \Rightarrow \quad (b, a) > (d, c)$$

$$d > a \vee c \quad \Rightarrow \quad (d, c) > \mathbf{(a, \varnothing)}$$

contrary to transitivity of the contrasts.

Assumption 1 is a necessary condition for the proposition ordering to be probability agreeing. It can be realized by subtraction of agreeing probability values when they exist.

From the results of the previous section, if Assumption 1 is granted, then there necessarily exists a transitive partial ordering of same-size bags of contrasts. The ordered contrast-bags can help us to enrich the partial order of proposition-bags.

Notation. A bag of propositions may be denoted as an indexed propositional variable enclosed in square brackets, as $[u_i]$. A bag of contrasts may be denoted as a contrast of indexed propositional variables enclosed in square brackets, as in $[(u_i, v_i)]$. The device $[(\varnothing, \varnothing), ...]$ denotes a bag of contrasts, all of whose elements are $(\varnothing, \varnothing)$, of whatever size is appropriate for the context in which the device appears. If some relationship is asserted between a contrast-bag C and $[(\varnothing, \varnothing), ...]$, then the size of the bag denoted by the device is the size of C.

Suppose we are presented with two same-sized proposition-bags, Q and R. We are asked a narrow question: whether or not we see any advantage of one over the other regarding the number of propositions that would be found true, if and when our uncertainty is resolved.

Of course, if there is some pairing of the propositions in the bags that allows us to assert an object-matching ordering between bags, $Q > R$, $R > Q$, or $Q = R$, then we would have our answer. Put another way, the question is simple if there is some indexing scheme over the propositions where the bag of contrasts between corresponding propositions in Q and R, $[\ (\ q_j,\ r_j\ )\ ]$, is ordered by object matching relative to the bag denoted by $[\ (\ \varnothing, \varnothing\ ), \ldots\ ]$.

Suppose there is no such pairing. The bags Q and R do not participate in an object-matching partial order.

Suppose further that there were two other proposition-bags, S and T, of the same size as Q and R, and for these bags, there is an object-matching order, $S = T$. There are also matching schemata for the propositions in each of $\{\ Q, R\ \}$ and $\{\ S, T\ \}$ where $[\ (\ q_k,\ r_k\ )\ ] = [\ (\ s_i,\ t_i\ )\ ]$, in the object matching sense.

Given the role that contrasts play in credal judgments, these circumstances provide a defensible basis for thinking that there may be no advantage between Q and R, just as we think there is no advantage between S and T. We might answer the narrow question about them in the negative.

Now suppose somewhat different circumstances obtain. We notice the situation among Q, R, S, and T just discussed, but we also think that $Q > R$ in the object-matching sense. That is, we find ourselves claiming an advantage for Q over R, in the face of a reason to think that there is no such advantage. This dissonance would be an excellent justification for a re-examination of how our felt beliefs ought to inform our answer to the question asked.

To be formal:

Definition. If Assumption 1 holds, then a relation between contrast-bags, denoted by ~, is asserted as follows: $[u_i] = [v_i] \Rightarrow [\ (\ u_i,\ v_i\ )\ ] \sim [\ (\ \varnothing, \varnothing\ ), \ldots\ ]$, where i is an index, arbitrarily attached to the propositions in each proposition-bag for notational discrimination, and where the u_i and v_i are propositions, and for same-sized contrast-bags X and Y, with "=" being the contrast-matching partial order's equal ranking, $X = Y \sim [\ (\ \varnothing, \varnothing\ ), \ldots\ ]$ implies $X \sim [\ (\ \varnothing, \varnothing\ ), \ldots\ ]$.

Assumption 2. There is a transitive partial ordering of same-size bags of propositions, denoted by the relational operators $\{\ >^*,\ =^*\ \}$, in which $[\ (\ u_i,\ v_i\ )\ ] \sim [\ (\ \varnothing, \varnothing\ ), \ldots\ ] \Rightarrow [\ u_i\ ] =^* [\ v_i\ ]$, and for proposition-bags X and Y, $X > Y \Rightarrow X >^* Y$.

As defined, the *tilde* relation distinguishes only two categories of contrast-bags: those that arise from equally ranked proposition-bags (regardless of how the propositions are paired to form the contrasts), and contrast-bags that are ranked equally with those of the first category. It is easy to verify that in neither category can contrast-bags be strictly ordered with respect to $[\ (\ \varnothing, \varnothing\ ), \ldots\ ]$ under Assumption 1, since that would imply a strict ordering between the Definition's $[u_i]$ and $[v_i]$, contrary to hypothesis and the Proposition of Section 3.

The motivation of Assumption 2 is to describe the application of essentially the same trade-off reasoning which decides the order of two propositions to the question of whether two portfolios might have differently attractive prospects for how many of their propositions might be found true. Specifically, the assumption licenses an inference from the existence of a *tilde* relationship that the relevant proposition-bags are ranked equally in a transitive partial ordering which extends the object-matching partial order.

The assumption allows the believer to have and to defend opinions about parity (or lack of advantage) between portfolios whose propositions do not exhibit pairwise equality. That is a respectable cognitive task in its own right. One might have assumed even more on the same intuition, but this much assumption is enough for the purpose at hand.

Assumption 2 is a necessary condition for probability agreement. It can be realized by addition over the agreeing probabilities or their differences in a bag's contents.

Theorem. A complete, definite, bounded, and transitive ordering of propositions has an agreeing probability density if Assumptions 1 and 2 hold.

A sketch proof of the Theorem appears in the Appendix.

5 Conclusions

Although Scott made no normative claims for his characterization of the key sufficient condition for probability agreement, it is nevertheless susceptible of respectable normative interpretations, both with and without a gambling element. Enriching de Finetti's ordinal insight with judgments about lesser or more pronounced inequality (including provision for equality) leads to other kinds of arguments, perhaps closer in spirit to de Finetti's conception of the original conjecture.

Throughout the paper, results have been presented for complete, definite orderings of propositions. Partial orderings and indefinite orderings (*i.e.* where $A \geq B$ might be asserted, but neither $A > B$ nor $A = B$) may realistically portray some defensible states of belief. The chief results here are immediately adaptable for partially and indefinitely ordered propositions, since these orderings are weaker than complete definite ones. In many contexts, it would be acceptable simply to assume outright that any ordering has some regulated definite completion on the same domain.

Belief change and other aspects of qualitative conditional probabilities were not discussed here. That is because other authors have established that if probability agreement for static or unconditional beliefs is secure, then solid motivations for the rest can be built upon that foundation. De Groot (1970) develops this theme for a variety of statistical applications, and gives pointers to related work.

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Appendix

Sketch Proof of the Proposition (section 3). If two bags are definitely ordered, then the items in the associated lists of elements sorted descendingly (with ties broken arbitrarily) are definitely ordered, i-th item paired with i-th item, in the required senses, all ≥ and some > for strict bag order, all equal for equal bag order. There can be only one suite of pairwise ordering assertions for the two fixed lists, and so at most one definite order of the bags.

Proof of the list ordering claim is by induction. Obviously, the Proposition is true for bags of one object each. Suppose it is true for sorted list of size m. Add to these one more object apiece, *high-object* ≥ *low-object*, where the *high-object* is ranked j among its peers in one list (the *high* list), and *low-object* is ranked k in the other list (the *low* list). By cases:

If $j = k$, then the proposition holds.

If index $j <$ index k: the j-th high element ≥ the j+1st high element ≥ the j-th low element, and similarly through index k-1. The k-th high element ≥ the k-1st low element ≥ the k-th low element. Elements ranked before j or after k, if any, are paired as among the m.

If $j > k$: The k-th high element ≥ the j-th high element ≥ the k-th low element; for indices i from k+1 through j, the i-th high element ≥ the j-th high element ≥ the k-th low element ≥ the i-th low element. Elements ranked before j or after k, if any, are paired as among the m.

Similar considerations show that if there is any strict inequality among the displaced elements (including the new additions), then at least one strict inequality will emerge, and if there is no such strict inequality, then none will be introduced. //

Sketch Proof of the Theorem (section 4). Suppose X and Y are proposition-bags of the same size, m, $X > Y$, and among the propositions, each atom appears the same number of times in both bags. Each bag has n atoms present in all. Let Q be the bag containing n-m ∅'s. Let A be a bag whose elements are n atomic propositions, in which each atom appears the same number of times as it does in X and in Y. By concatenation, X & $Q > Y$ & Q, so X & $Q >^* Y$ & Q. If Assumptions 1 and 2 hold, however, X & $Q =^* A =^* Y$ & Q, so X & $Q =^* Y$ & Q.

The method of demonstrating the equal ranking with A iterates the process illustrated here for one proposition of three atoms, $a \vee b \vee c$, which is placed in a bag along with two ∅'s, and it is shown that that bag =* [a, b, c], the bag of $a \vee b \vee c$'s constituent atoms.

[(a ∨ b ∨ c, a), (∅, b ∨ c), (∅, ∅)] = [(b ∨ c, ∅), (∅, b ∨ c), (∅, ∅)] by element matching, since $a \vee b \vee c = a \vee b \vee c$ implies ($a \vee b \vee c$, a) = ($b \vee c$, ∅) by Assumption 1. From [b ∨ c, ∅, ∅] = [b ∨ c, ∅, ∅], [(b ∨ c, ∅), (∅, b ∨ c), (∅, ∅)] ~ [(∅, ∅), ...], by definition of tilde, and also [(a ∨ b ∨ c, a), (∅, b ∨ c), (∅, ∅)] ~ [(∅, ∅), ...], so by Assumption 2, [a ∨ b ∨ c, ∅, ∅] =* [a, b ∨ c, ∅].

Similarly, we have [(a, a), (b ∨ c, b), (∅, c)] = [(a, a), (c, ∅), (∅, c)] ~ [(∅, ∅), ...], so [a, b ∨ c, ∅] =* [a, b, c]. By transitivity, [a ∨ b ∨ c, ∅, ∅] =* [a, b, c].

The procedure clearly generalizes to disjunctions of any finite length, and can be performed on bags of propositions of any finite size. X & Q and Y & Q lead to the same all-atomic bag A, since they have the same population of atoms. So, any violation of the key sufficient condition identified by Kraft, Pratt, and Seidenberg and Scott corresponds to a defeat of the Assumptions. //