S-Independence and S-Conditional Independence with respect to Upper and Lower Conditional Probabilities Assigned by Hausdorff Outer and Inner Measures

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Abstract

In this paper the notion of s-irrelevance with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures is proved to be a sufficient condition for strong independence introduced for credal sets. An example is given to show that the converse is not true. Moreover the definition of s-conditional irrelevance is given and a generalized factorization property is proposed as necessary condition of s-conditional irrelevance. Examples are given to show that s-conditional irrelevance and s-irrelevance are not related; moreover sufficient conditions are given for equivalence between s-conditional irrelevance and s-irrelevance. Finally the notion of s-irrelevance is extended to random variables.

Keywords. Independence, Strong independence Conditional independence, Hausdorff outer and inner measures.

1 Introduction

Coherent upper and lower conditional probabilities in the sense of Walley ([17]) are required to be *separately coherent*, that is, for every conditioning event B the restrictions $\overline{P}(\cdot|B)$ and $\underline{P}(\cdot|B)$ are upper and lower probabilities and $\overline{P}(B|B)=1$ and $\underline{P}(B|B)=1$.
This property is not always satisfied, in a continuous framework, if upper and lower conditional probabilities are obtained as natural extensions of a conditional probability defined in the axiomatic way. This is due to some problems related to the axiomatic definition of regular conditional probability (see [2], [10], [15], [11]).

To avoid this problem in [11] coherent upper and lower conditional probabilities are given as *natural extensions* of a finitely additive conditional probability in the sense of Dubins. These coherent upper and lower conditional probabilties arise through the use of Hausdorff outer and inner measures. From the axioms defining finitely conditional probabilities in the sense of Dubins it follows that upper and lower conditional probabilities defined by Hausdorff outer and inner measures are separately coherent.

A concept related to the definition of probability and conditional probability is probabilistic independence. In a continuous probabilistic space $(\Omega,\mathbf{F},P)$, where probability is usually assumed equal to the Lebesgue measure we have that the finite, countable and fractal sets (i.e. the sets with Hausdorff dimension non integer) have probability equal to zero. For these sets the standard definition of independence, given by the factorisation property, is always satisfied since both members of the equality are zero. Moreover the notions of *epistemic irrelevance* and *epistemic independence*, proposed by Walley, are not related to the notion of logical independence when the events have zero lower probability (see [11]).

In [11] the notions of *s-irrelevance* and *s-independence* with respect to upper and lower conditional probabilities assigned by a class of Hausdorff outer and inner measures are proposed to test independence of sets with dimension less then the dimension of Ω and to assure that logical independence is a necessary condition of probabilistic independence.

The notions of s-irrelevance and s-independence are based on the concepts of *epistemic irrelevance* and *epistemic independence* of Walley with the furhter condition that events A and B and their intersection AB have the same Hausdorff dimension.

According to this approach to independence, sets that represent events can be imagined divided in different layers; in each layer there are sets with the same Hausdorff dimension; two events A and B are s-independent if and only if the events A and B and their intersection AB belong to the same layer and they are *epistemically independent*.

Moreover in ([11]) it has been proved that s-irrelevance implies logical independence.

In Section 3 of this paper the link between *s-independence* and of *strong independence* proposed by Levi ([14]), for *credal sets* of probabilities, is investigated. This property essentially requires that each extreme point of the credal set K(A∩B) satisfies the the factorization property with the marginal K(A) and K(B).
It is proved that s-irrelevance implies strong independence and an example is given to show that the converse is not true. Moreover examples of s-independent events are given.
In Section 4 the definitions of *s-conditional irrelevance* and s-*conditional independence* are proposed and the factorization property is generalized to the case where the conditioning event is different to Ω. It is proved that the *generalized factorization proper*ty is a necessary condition of s-conditional irrelevance. An example is given to show that the convere is not true.
Moreover s-conditional irrelevance and s-irrelevance are compared. In general the two concepts are not related; events A, B and C are proposed, such that B is s-conditional irrelevance to A given C, but B is not s-irrelevance to A.
Moreover events A and B are considered such that B is s-irrelevant to A but B is not s-conditional irrelevant to A given Ω.
It is proved that the two notions are equivalent when C is equal to Ω and the events A, B, AB and the complement of B have Hausdorff dimension equal to Ω.

2 Upper and Lower Conditional Probabilities Assigned by a Class of Hausdorff Outer and Inner Measures

Coherent upper and lower conditional probabilities, in the approach proposed by Walley [17], are a special case of coherent upper and lower conditional previsions $\overline{P}(X|B)$ and $\underline{P}(X|B)$ that are characterized in the case where conditioning events B form a partition **B** of Ω and X are 0-1valued gambles.
Let **F** be the σ-field of all subsets of Ω=[0,1] and let **B** be the partition of Ω that consists of all singletons of [0,1]. For each $\{\omega\}$ in **B** and A in **F** $\overline{P}(A|\{\omega\})$ and $\underline{P}(A|\{\omega\})$ are *separately coherent* when for every conditioning event $\{\omega\}$ $\overline{P}(\cdot|\{\omega\})$ and $\underline{P}(\cdot|\{\omega\})$ are coherent upper and lower probability on **F** and $\overline{P}(\{\omega\}|\{\omega\})=1$ and $\underline{P}(\{\omega\}|\{\omega\})=1$.
In the axiomatic approach conditional probability is defined with respect to a σ-field **G** of conditioning events by the Radon-Nikodym derivative; the two definitions can be compared when the σ-field **G** is generated by the partition **B**.
In [11] it has been proved that every time that the σ-field **G** of conditioning events is properly contained in **F** and it contains all singletons of [0,1] then conditional probability defined by the Radon-Nikodynm derivative is not separately coherent.
So in this case upper and lower conditional probability can not be obtained as extensions to the class of all subsets of Ω of a conditional probability defined in the axiomatic way.
An alterative approach that assures to conditional probability the property to be separately coherent is that one proposed by Dubins [12].
In this section upper and lower conditional probabilities are obtained as *natural extensions* (Theorem 3.1.5 [17]) of a finitely additive conditional probability in the sense of Dubins .
Let Ω a non empty set and let **F** and **G** be two fields of subsets of Ω, with **G**⊆**F** or with **G** an additive subclass of **F** , P* is a *finitely additive conditional probability* ([12]) defined on **(F,G)** if it is a real function defined on **F**×$\mathbf{G}^0$, where $\mathbf{G}^0=\mathbf{G}-\varnothing$, such that the following conditions hold:

I) given any $H\in\mathbf{G}^0$ and $A_1,\dots,A_n\in\mathbf{F}$ with $A_i\cap A_j=\varnothing$ for $i\neq j$, the function P*(·|H) defined on **F** is such that

I) $P^*(A|H)\geq 0$, $\quad P^*(\bigcup_{k=1}^{n} A_k|H)=\sum_{k=1}^{n} P^*(A_k|H)$,

$P^*(\Omega|H)=1$

II) $P^*(H|H)=1$ if $H\in \mathbf{F}\cap\mathbf{G}^0$

III) given $E\in\mathbf{F}$, $H\in\mathbf{F}$ $EH\in\mathbf{F}$ with $A\in\mathbf{G}^0$ and $EA\in\mathbf{G}^0$ then $P^*(EH|A)=P^*(E|A)P^*(H|EA)$.

From conditions I) and II) we have

II') $P^*(A|H)=1$ if $A\in\mathbf{F}$, $H\in\mathbf{G}^0$ and $H\subset A$.

Such approach to conditional probability allows to give probability assessments on arbitrary finite family of conditional events through the notion of *coherence* as proposed by de Finetti ([7], [8]). In fact, if **F** and **G** are arbitrary finite families of subsets of Ω, then the real function P, defined on **F**×$\mathbf{G}^0$ is *coherent* if and only if it is the restriction of a finitely additive conditional probability defined on **D**×$\mathbf{D}^0$, where **D** is the field generated by the sets of **F** and **G**.
In [10] a finitely additive conditional probability in the sense of Dubins is defined by a class of Hausdorff dimensional measures.

2.1 Preliminaries about Hausdorff Outer and Inner Measures

Let (Ω,d) be the Euclidean metric space with $\Omega=[0,1]$. The diameter of a nonempty set U of Ω is defined as $|U|=\sup\{|x-y|: x,y\in U\}$ and if a subset A of Ω is such that $A\subset\bigcup_i U_i$ and $0<|U_i|<\delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A. Let s be a non-negative number. For $\delta>0$ we define $h^s_\delta(A)=\inf\sum_{i=1}^{\infty}|U_i|^s$, where the infimum is over all δ-covers $\{U_i\}$. The Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$, is defined as $h^s(A)=\lim_{\delta\to 0} h^s_\delta(A)$. This limit exists, but may be infinite, since $h^s_\delta(A)$ increases as δ decreases.

The *Hausdorff dimension* of a set A, $\dim_H(A)$, is defined as the unique value, such that

$$h^s(A)=\begin{cases}\infty & \text{if } 0\le s<\dim_H(A)\\ 0 & \text{if } \dim_H(A)<s<\infty\end{cases}$$

We can observe that if $0<h^s(A)<\infty$ then $\dim_H(A)=s$, but the converse is not true. We assume that the Hausdorff dimension of the empty set is equal to -1 so no event has Hausdorff dimension equal to the empty set.

If an event A is such that $\dim_H(A)=s<1$ than the Hausdorff dimension of the complementary set A^c is equal to 1 since the following relation holds:

$$\dim_H(A\cup B)=\max\{\dim_H(A);\dim_H(B)\}.$$

A subset A of Ω is called measurable with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E)=h^s(A\cap E)+h^s(E-A)$ for all sets $E\subset\Omega$.

The restriction of h^s to the σ-field of h^s- measurable sets, containing the σ-field of the borelian sets, is called *Hausdorff s-dimensional measure*. In particular the Hausdorff 0-dimensional measure is the counting measure and the Hausdorff 1-dimensional measure is the Lebesgue measure.

The most familiar set of real numbers of non-integer Hausdorff dimension is the Cantor set.

Let $E_0=[0,1]$, $E_1=[0,1/3]\cup[2/3,1]$, $E_2=[0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]$,etc., where E_{j+1} is obtained by removing the open middle third of each interval in E_j

The Cantor's set is the perfect set $E=\bigcap_{j=0}^{\infty}E_j$ .The Hausdorff dimension of the Cantor set is s=log2/log3 and $h^s(E)=1$. The Cantor set and its complementary set will be considered in Example 4 of Subsection 4.2.

For more details about Hausdorff measures see, for example Falconer ([13]).

2.2 A New Model of Upper and Lower Conditional Probabilities

In [10] [11] upper (lower) conditional probability is given by Hausdorff s-dimensional outer (inner) measure if the conditioning event has positive and finite Hausdorff s-dimensional outer measure; otherwise upper conditional probability is defined by a 0-1 finitely additive (but not countable additive) probability so that condition III) of a finitely additive conditional probability in the sense of Dubins is satisfied.

Theorem 1. *Let $\Omega=[0,1]$, **F** is the σ-field of all subsets of Ω and let **G** be an additive sub-class of **F**. Let us denote by h^s the Hausdorff s-dimensional outer measure and let define on $\mathbf{C}=\mathbf{F}\times\mathbf{G}^0$ the function $\overline{P}$ by*

$$\overline{P}(A|H)=\begin{cases}\dfrac{h^s(A\cap H)}{h^s(H)} & \text{if } 0<h^s(H)<\infty\\ m(A\cap H) & \text{if } h^s(H)=0,\infty\end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure. Then the function $\overline{P}$ is an upper conditional probability.

The existence of the measure m is a consequence of the prime ideal theorem.

The coniugate lower conditional probability $\underline{P}$ can be defined as in Theorem 1 if h^s denotes the Hausdorff s-dimensional inner measure.

Let **B** the partition of all singletons $\{\omega\}$of Ω. The functions $\overline{P}(\cdot|\{\omega\})$ and $\underline{P}(\cdot|\{\omega\})$ are *separately coherent*, in the sense of Walley, in fact they are respectively upper and lower coherent probability on **F** and $P(\{\omega\}|\{\omega\})=1$.

3 S-independence

In [11] the notions of s-irrelevance and s-independence have been introduced with the aim to assure that logical independence is a necessary condition of stochastic independence.

Two events A and B are logically independent if the four sets $A\cap B$, $A\cap B^c$, $A\cap B^c$, $A^c\cap B^c$ are non-empty.

In a continuous probability space, where probability is usually defined by the Lebesgue measure on [0,1], logical independence and stochastic independence are non related. In fact events represented by finite or

countable sets, fractal sets (i.e. sets with non integer Hausdorff dimension) always satisfy the standard definition of stochastic independence given by the factorization property, that is $P(A\cap B)=P(A)P(B)$ even if they are logically dependent.
Also the notion of independence with respect to a σ-field ([1]) and the concepts of epistemic irrelevance and epistemic independence ([17]) are not related to the logical independence (see Example 1, Example 2, Example 3 of [11]).
In this section other aspects of s-independence are investigated; in subsection 3.1 we compare the notion of s-irrelevance and s-independence with the notion of *strong independence* proposed by Levi ([14]).
In subsection 3.2 some examples of s-independent events are given.

3.1 S-Independence and Strong Independence

The notions of *s-irrelevance* and *s-independent* are based on the concepts of epistemic irrelevance and epistemic independence proposed by Walley ([17]) with the further condition that the relative events and their intersection must have the same Hausdorff dimension;

When the events A and B or their complements have not upper probability equal to zero, epistemic independence implies logical independence. Otherwise we can have that logically dependent events can be epistemically independent.

Example 1. $\Omega=[0,1]$, let **F** be the σ-field of all subsets of [0,1] and let **G** be the additive sub-class of **F** of sets that are finite and co-finite. Let A and B two finite subsets of [0,1] such that $A\cap B=\emptyset$. If conditional probability is defined as in Theorem 2 we have that

$$\underline{P}(A|B)=\overline{P}(A|B)=\frac{h^0(A\cap B)}{h^0(B)}=0;$$

$$\underline{P}(A|B^c)=\overline{P}(A|B^c)=\frac{h^1(A\cap B^c)}{h^1(B^c)}=0 \quad \text{and}$$

$$\underline{P}(A)=\overline{P}(A)=\overline{P}(A|\Omega)=\frac{h^1(A)}{h^1(\Omega)}=0$$

So A and B are logical dependent but epistemically independent.

The previous example put in evidence the necessity to introduce the following definition.

Two events A and B are s-independent if they and their intersection have the same Hausdorff dimension and they are epistemically independent with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures.

Definition 1. *Let $\Omega=[0,1]$, let **F** be the σ-field of all subsets of Ω and let be **G** an additive subclass of **F**. Denoted by $\overline{P}$ and $\underline{P}$ the upper and lower conditional probabilities defined by the Hausdorff outer and inner measures and given A in **F** and B and C in $\boldsymbol{G}^0$, then B is s-irrelevant to A if the following conditions holds*

1) $\dim_H(A\cap B)=\dim_H(A)=\dim_H(B)$

2) $\overline{P}(A|B)=\overline{P}(A|B^c)=\overline{P}(A|\Omega)$ *and*

$\underline{P}(A|B)=\underline{P}(A|B^c)=\underline{P}(A|\Omega)$.

Definition 2. *Let $\Omega=[0,1]$, let **F** be the σ-field of all subsets of Ω and let be **G** an additive subclass of **F**. Denoted by $\overline{P}$ and $\underline{P}$ the upper and lower conditional probabilities defined by Hausdorff outer and inner measures and given A in **F** and B in $\boldsymbol{G}^0$, then A and B are s-independent if B is s- irrelevant to A and A is s-irrelevant to B.*

The notions of s-irrelevance and s-independence are compared with the concept of *strong independence* given for credal sets of probabilities ([14]).
A non-empty set K of probability measures is called a *credal set*; assuming that all the probabilities in K are defined on the same algebra, we can associate to any event A belonging to this algebra a set of number denoted by K(A) determined by the values assumed by the probability measures of K in A.
In particular given a countable additive probability measure P and an event A belonging to the σ-field that is the domain of P, the *natural extensions* of P are the inner and outer measures generated by P (Walley Theorem 3.1.5); they determine the largest set associate to A, that is $K(A)=[\underline{P}(A);\overline{P}(A)]$.
Two events A and B are *strongly independent* when every extreme point of $K(A\cap B)$ satisfies the standard definition of stochastic independence, given by the factorization property.
Given two sets K(A) and K(B) there may be several sets $K(A\cap B)$, called *extension* of K(A) and K(B), for which A an B are independent; the *strong extension* is the largest joint set $K(A\cap B)$ satisfying strong independence with K(A) and K(B).
If $K(A)=[\underline{P}(A);\overline{P}(A)]$ and $K(B)=[\underline{P}(B);\overline{P}(B)]$ and the factorization properties with respect to lower and upper probabilities hold , that is

$\underline{P}(A\cap B)=\underline{P}(A)\underline{P}(B)$ and $\overline{P}(A\cap B)=\overline{P}(A)\,\overline{P}(B)$,

then A and B are strongly independent and their strong extension is $K(A\cap B)=[\underline{P}(A\cap B),\ \overline{P}(A\cap B)]$.
The following result proves that if an event B is s-irrelevant to an event A with respect to upper and lower

conditional probabilities assigned by Hausdorff outer and inner measures then A and B are strongly independent.

Theorem 2. *Let Ω=[0,1] , let **F** be the σ-field of all subsets of Ω and let be **G** additive subclass of **F**. Denoted by $\overline{\mathrm{P}}$ and $\underline{\mathrm{P}}$ the upper and lower conditional probabilities defined by the Hausdorff outer and inner measures and given A in **F** and B in $\mathbf{G}^0$, we have that if B is s-irrelevant to A then the upper and lower conditional probabilities $\overline{\mathrm{P}}$ and $\underline{\mathrm{P}}$ satisfy the factorization property.*

Proof. We prove that $\overline{\mathrm{P}}$ satisfies the factorization property. The same reasoning can be use to prove that also $\underline{\mathrm{P}}$ satisfies the factorization property.

Recalling that $\overline{\mathrm{P}}(A)=\overline{\mathrm{P}}(A|\ \Omega)$, different cases are considered:

a) if $\dim_H(B)<1$ the factorization properties $\overline{\mathrm{P}}(A\cap B)=\overline{\mathrm{P}}(A)\ \overline{\mathrm{P}}(B)$ is satisfied since it vanishes to 0=0.

b)) if $\dim_H(B)=1$ and $h^1(B)>0$, from condition 2) of the definition of s-irrelevance we have $\frac{h^1(A\cap B)}{h^1(B)}=h^1(A)$, that is the factorization property.

c) if $\dim_H(B)=1$ and $h^1(B)=0$ the factorization property becomes

$$h^1(A\cap B)=h^1(A)\,h^1(B)$$

and it is satisfied since it vanishes to 0=0. □

Remark 1. The converse of the Theorem 2 is not true since no condition about the Hausdorff dimension of the sets that represent the events is given in the definition of strong independence. Moreover even if the sets A, B and their intersection A∩B have the same Hausdorff dimension the factorization property does not imply s-irrelevance since this is not a symmetric notion.

Example 2. Let Ω=[0,1], let A be a finite set, B=[a,b] with $0\leq a<b\leq 1$. Recalling that $\overline{\mathrm{P}}(A)=\overline{\mathrm{P}}(A|\Omega)$ and $\underline{\mathrm{P}}(A)=\underline{\mathrm{P}}(A|\ \Omega)$ and that A and B are measurable with respect to the Hausdorff measure of order 1 h^1 (so upper and lower conditional probability are equal), we have that the credal sets K(A) and K(B) are singletons; moreover the factorization property is satisfied because it vanishes to 0=0. Then A and B are strongly independent but they are not s-independent, in fact condition 1 of the definition of s-irrelevance is not satisfied since $\dim_H(A)=0$ while $\dim_H(B)=1$.

In Cozmann ([4]), the Kuznetsov's condition has been introduced as a tool to construct credal sets and to analysing independence between random variables.

Given two random variables X and Y this condition is given with respect to upper and lower expectation of f(X) and g(Y) where f and g are any bounded functions.

Kuznetsov's condition is not equivalent to strong independence as shown in Example 1 of [3]. But when the credal sets K(X) and K(Y), that are the credal sets defined respectively by a collection of density p(X) and p(Y), are singletons then there is a single joint probability density that satisfied strong independence and Kuznetsov's condition.

Since the problem to define upper and lower expectation with respect to Hausdorff outer and inner measures is not yet analyzed then we can compare the notion of s-irrelevance with the Kuznetsov's condition of independence only in the case where the random variables X and Y are indicator functions of events.

In particular if X and Y are respectively the indicator function of the sets A and B of the previous Example 2 the variables X and Y are strong independent, they satisfy the Kuznetsov's independence condition , but B is not irrelevant to A .

In the paper of Couso, Moral and Walley [4] other notions of independence are introduced when Ω is a finite set and all marginal probabilities are non zero. In this case upper and lower conditional probabilities defined by Hausdorff outer and inner measures are given by the counting measure, that is the Hausdorff measure of order 0 since all events are finite sets. We have that in this case, the notion of s-irrelevance with respect to Hausdorff outer and inner measures and strong independence are equivalent for all compatible events since condition 1) of s-irrelevance is always satisfied because all events have Hausdorff dimension equal to zero.

Examples given in [4] compare different notions of independence when all events have the Hausdorff dimension equal to zero; the concepts of s-irrelevance and s-independence can be useful when we have to study independence for events that have different Hausdorff dimension.

3.1 Examples of S-Independent Events

If upper and lower conditional probabilities are defined respectively by Hausdorff outer and inner measures then we have that:

-every event B is not s-irrelevant for ∅ since condition 1) of the previous definition is never satisfied;

-every event B such that $\dim_H(B)=\dim_H(\Omega)$ is s-irrelevant for Ω;

-every countable set B is s-irrelevant to a finite set A such that A∩B≠∅, but A and B are not s-independent since A is not s-irrelevant to B.

In the definitions of s-irrelevance for events A and B no condition is given on the Hausdorff dimension of their complementary sets; in fact two events can be s-independent even if their complementary sets have different Hausdorff dimension.

Example 3. Given A=$\left[0,\frac{1}{2}\right]$ and B= [0,1]-$\left\{\frac{1}{2},\frac{2}{3}\right\}$ we have that B is s-irrelelvant to A. In fact the events A, B and AB have Hausdorff dimension equal to 1 so that condition 1) of the definition of s-irrelevance is satisfied ; moreover condition 2) becomes

$$\frac{h^1(A\cap B)}{h^1(B)}=\frac{h^0(A\cap B^c)}{h^0(B^c)}=h^1(A)$$

and it is satisfied since it vanishes to $\frac{1}{2}=\frac{1}{2}=\frac{1}{2}$.

We can also observe that A is-irrelevant to B since condition 2) becomes

$$\frac{h^1(B\cap A)}{h^1(A)}=1\frac{h^1(B\cap A^c)}{h^1(A^c)}=h^1(B)$$

and it is satisfied since it vanishes to 1=1=1.
So A and B are s-independent.

4 S-Conditional Independence

The notions of s-irrelevance and s-independence with respect to upper and lower conditional probabilities assigned by a class of Hausdorff outer and inner measures can be useful when you have to study independence for events that have different Hausdorff dimension (i.e. finite or countable sets, interval or fractal sets).

In the previous section the factorization property of upper and lower conditional probabilities defined respectively by Hausdorff outer and inner measures, has been proved when an event B is s-irrelevant to another event A.

Theorem 2 of the previous Section proves that the factorization property is satisfied when the events A, B, A∩B and Ω=[0,1] have the same Hausdorff dimension and B is s-irrelevant to A.

We can also observe that if the events A, B, A∩B have Hausdorff dimension less than that one of Ω, the factorization property is obviously satisfied because it vanishes to 0=0. This happens because the factorization property is verified with respect to the outer (or inner) measure of order 1, that is the Hausdorff dimension of Ω.

We want to investigate if, in this case, a more general factorization property with respect to a conditioning event C is also satisfied. Denoted by s the Hausdorff dimension of a conditioning event C we want to investigate when a *generalized factorization property* is satisfied with respect to Hausdorff outer and inner s-dimensional measures.

With this aim a *generalized factorization property,* with respect to any conditioning event C is introduced and it is proved to be a necessary condition of the notion of s-conditional irrelevance.

4.1 S-Conditional Irrelevance and the Generalized Factorization Property

The notion of *s-conditional irrelevance* and *s-conditional independence* are introduced with respect to upper and lower conditional probabilities assigned by Hausdorff outer or inner measures.

Definition 3. *Let Ω=[0,1], let **F** be the σ-field of all subsets of Ω and let be **G** additive subclass of **F**. Denoted by* $\overline{P}$ *and* $\underline{P}$ *the upper and lower conditional probabilities defined by Hausdorff outer and inner measures and given A in **F** and B and C in* $\boldsymbol{G}^0$*, then B is s-conditional irrelevant to A given C if the following conditions holds*

1a)

$dim_H(A\cap B\cap C)=dim_H(B\cap C)=dim_H(A\cap B^c\cap C)=$

$dim_H(B^c\cap C)= dim_H(A\cap C)= dim_H(C)$

2a) $\overline{P}(A|B\cap C) = \overline{P}(A|B^c\cap C) = \overline{P}(A|C)$ *and*

$\underline{P}(A|B\cap C)=\underline{P}(A|B^c\cap C)= \underline{P}(A|C)$

We can observe that if $\dim_H(B^c)$=1 then condition 1a) is equivalent to condition

1a') $dim_H(A\cap B\cap C)= dim_H(A\cap C)= dim_H(B\cap C)= dim_H(C)$

Definition 4. *Let Ω=[0,1] , let **F** be the σ-field of all subsets of Ω and let be **G** additive subclass of **F**. Denoted by* $\overline{P}$ *and* $\underline{P}$ *the upper and lower conditional probabilities defined by Hausdorff outer and inner measures and given A in F and B and C in* $\boldsymbol{G}^0$*, then A and B are s-conditional independent given C if B is s-conditional irrelevant to A given C and A is s-conditional irrelevant to B given C.*

We are interested now to generalize the factorization property, in the case where the conditioning event is not necessary Ω.

Definition 5. *Let Ω=[0,1] , let **F** be the σ-field of all subsets of Ω and let be **G** additive subclass of **F**. Denoted by* $\overline{P}$ *and* $\underline{P}$ *the upper and lower conditional probabilities defined by Hausdorff outer and inner measures and given A in **F** and B and C in* $\boldsymbol{G}^0$*, we say* *that* $\overline{P}$ *and* $\underline{P}$ *satisfy the generalized factorization property if the following equalities hold:*

$$\overline{P}(A\cap B|C)= \overline{P}(A|C)\,\overline{P}(B|C) \text{ and}$$

$$\underline{P}(A\cap B|C)= \underline{P}(A|C)\,\underline{P}(B|C)$$

Next theorem proves that the generalized factorization property is a necessary condition of the notion of s-conditional irrelevance.

Theorem 3. *Let Ω=[0,1] , let* ***F*** *be the σ-field of all subsets of Ω and let be* ***G*** *additive subclass of* ***F****. Denoted by* $\overline{P}$ *and* $\underline{P}$ *the upper and lower conditional probabilities defined by Hausdorff outer and inner measures and given A in* ***F*** *and B and C in* $\boldsymbol{G}^0$ *such that B is s-conditional irrelevant to A given C then* $\overline{P}$ *and* $\underline{P}$ *satisfy the generalized factorization property.*

Proof. We prove that $\overline{P}$ satisfies the generalized factorization property. The same reasoning can be use to prove that also $\underline{P}$ satisfies the generalized factorization property.

Let $s=\dim_H(C)$; we have to consider the following cases:

a) if $0<h^s(C)<\infty$, $0<h^s(B\cap C)<\infty$ and $0<h^s(B^cC)<\infty$, since B is s-irrelevant to A given C from the definition of s-conditional irrelevance we have

$$\frac{h^s(A\cap B\cap C)}{h^s(B\cap C)}=\frac{h^s(A\cap B^c\cap C)}{h^s(B^c\cap C)}=\frac{h^s(A\cap C)}{h^s(C)}$$

that implies

$$h^s(A\cap B\cap C)h^s(C)=h^s(A\cap C)h^s(B\cap C)$$

and dividing by $[h^s(C)]^2$ we obtain $\overline{P}(A\cap B|C)=\overline{P}(A|C)\overline{P}(B|C)$;

b)if $0<h^s(C)<\infty$, $h^s(B\cap C)=0$ and $0<h^s(B^c\cap C)<\infty$, the generalized factorization property is verified since it vanishes to 0=0.

b) if $0<h^s(C)<\infty$, $h^s(B^cC)=0$ and $0<h^s(B\cap C)<\infty$ then $h^s(C)=h^s(B\cap C)$ and from the definition of s-conditional irrelevance we have $\frac{h^s(A\cap B\cap C)}{h^s(B\cap C)}=\frac{h^s(A\cap C)}{h^s(C)}$. It follows that the generalized factorization property is satisfied;

c) if $h^s(C)=0$ then $h^s(B\cap C)=0$ and $h^s(B^c\cap C)=0$, moreover from the definition of s-conditional irrelevance we have m(A∩B∩C)=m(A∩B ^{c} ∩C)=m(A∩C). The generalized factorization property becomes m(A∩B∩C)=m(A∩C) m(B∩C) and it is verified since it vanishes to 0=0 or 1=1 according to the fact that m(A∩B∩C)is equal to 0 or 1;

d) if $h^s(C)=\infty$, $h^s(B\cap C)=\infty$ and $h^s(B^c\cap C)=\infty$ then from the definition of s-conditional irrelevance we have that m(A∩B∩C)= $m(B^c\cap C)$=m(A∩C); if m(A∩B∩C)=0 then the generalized factorization property is verified since it vanishes to 0=0 otherwise if m(A∩B∩C)=1 then for the monotony of m we have that m(B∩C)=1 and so the generalized factorization property is verified.

e) if $h^s(C)=\infty$, $h^s(B^c\cap C)<\infty$ and $h^s(B\cap C)=\infty$ from the definition of s-conditional irrelevance we have that m(A∩B∩C)=m(A∩C) then the factorization property becomes m(A∩B∩C)=m(A∩C) m(B∩C) and it is verified since it vanishes to 0=0 or 1=1 according to the fact that m(ABC)is equal to 0 or 1;

f) if $h^s(C)=\infty$, $h^s(B^c\cap C)=\infty$ and $h^s(B\cap C)<\infty$ from the definition of s-conditional irrelevance we have that $m(A\cap B^c\cap C)$=m(A∩C) and so m(A∩B∩C)=0; moreover from the axiom III of a finitely additive conditional probability we have that m(B∩C)=0 then the generalized factorization property is satisfied since it vanishes to 0=0. □

The generalized factorization property does not require any condition on the Hausdorff dimension of sets so in general it does not imply s-conditional irrelevance. In particular we prove that even if condition 1 of s-conditional irrelevance is satisfied the generalized factorization property does not imply s-conditional irrelevance. This follows also from the fact that s-conditional irrelevance is not a symmetric property as the generalised factorization property.

Example 4. Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and let **G** be a sub σ-field of **F**. Let us denote by $\overline{P}$ and $\underline{P}$ the upper and lower conditional probabilities defined by the Hausdorff outer and inner measures; given $A=\left\{\frac{1}{2},\frac{1}{4},\frac{1}{3}\right\}$, $B=\left\{\frac{1}{2}\right\}$ and C equal to the set of rationales of [0,1] we have that the generalized factorization property becomes m(A∩B∩C)=m(A∩C) m(B∩C) and it is verified since it vanishes to 0=0. Moreover denoted by P= $\overline{P}$ = $\underline{P}$ we have that the second condition of s-conditional irrelevance, $P(A \mid B \cap C) = P(A \mid B^c \cap C) = P(A \mid C)$, is not satisfied since $\frac{h^0(A \cap B \cap C)}{h^0(B \cap C)} = 1$ while

$m(A \cap B^c \cap C) = m(A \cap C) = 0$.

4.2 S-Conditional Irrelevance and S-Irrelevance.

In general the notion of s-conditional irrelevance is not related to the notion of s-irrelevance even if the conditioning event is Ω. An example of events s-independent but not s-conditional independent given Ω is given.

Example 5. Let A and B be respectively a finite and a countable subset of Ω=[0,1] with intersection different from the empty set. We have that B is s-irrelevant to A but B is not s-conditional irrelevant to A given Ω since condition 1a) is not satisfied because the Hausdorff dimension of C= Ω is 1 while the Hausdorff dimension of A, B and A∩B is 0.

Next result shows that the concepts of s-irrelevance and s-conditional irrelevance are equivalent when the event A, B, AB and B^c have the same Hausdorff dimension of C= Ω.

Theorem 4. *Given A, B and C subsets of Ω such that C= Ω=[0,1] and the Hausdorff dimension of A, B, A∩B and B^c is equal to 1, then B is s-conditional irrelevant to A given Ω if and only if B is s-irrelevance to A.*

Proof. Since the Hausdorff dimension of B^c is equal to 1 and B is s-conditional irrelevant to A given Ω then condition 1a') and 2a) are verified and they implies conditions 1) and 2) of the definition of s-irrelevance. □

In the following example events A, B and C are proposed such that B is s-conditional irrelevant to A given C but B is not s-irrelevant to A.

Example 6. Let Ω=[0,1] and let us denote $\overline{P}$ and $\underline{P}$ the upper and lower conditional probabilities defined by the Hausdorff outer and inner measures. Let A the complementary set of Cantor set (see subsection 2.1) , B=[0,1]- $\left\{\frac{1}{2}, \frac{1}{3}\right\}$ and C = $\left\{\frac{1}{2}, \frac{5}{12}\right\}$.

We have that B is not s-irrelevant to A since condition 2) is not satisfied, in fact

P(A|B)= $\frac{h^1(AB)}{h^1(B)} = 1$, P(A| B^c)= $\frac{h^0(AB^c)}{h^0(B^c)} = \frac{1}{2}$ and

P(A)= $h^1(A) = 1$

But B is s-conditional irrelevance to A given C. In fact condition 1a) is satisfied since

$\dim_H$ (ABC)= $\dim_H$ (BC)= $\dim_H$ (A B^cC)=

$\dim_H$ (B^cC)= $\dim_H$ (AC)= $\dim_H$ (C)=0

and also condition 2a) is verified because

P(A|BC)= $\frac{h^0(ABC)}{h^0(BC)} = 1$ = P(A| B^C C)=P(A|C).

We can observe that C is contained in A so we can say that if B is s-conditional irrelevance to A given C than A and C are not necessary logically independent.

5 S-Irrelevance for Random Variables

In this section the definition of s–irrelevance for random variables, is proposed and it is compared with the standard definition.

In Billingsley [1] the notion of independence for random variables is given in terms of σ-fields generated by them. This is due to the fact that conditional probability in the axiomatic approach is defined with respect to a σ-field of conditioning events. In the framework of coherent conditional probabilities this is not necessary.

A random variable is a function from Ω to R. Given a σ-field **G**, X is measurable with respect to **G** if the sets $\left\{\omega : X^{-1}(\omega) \in B\right\}$ belong to **G** for every borelian set B of R.

The σ-field σ(X), generated by a random variable X is the smallest σ-field with respect to which X is measurable, that is the intersection of all σ-fields with respect to which X is measurable. Two random variable X and Y are independent according to the definition given in [1] if the σ-fields σ(X) and σ(Y) generated by them are independent, that is, for each choice of A in σ(X) and B in σ(Y) the events A and B are independent, according to the standard definition of independence given by the factorization property.

As discussed in [9] the definition of independence given by the factorization property is not related to the notion of logical independence. So in this section the definition of s–irrelevance for random variables is proposed in terms of s-irrelevance of σ-fields generated by the random variable. Since these two classes of events are not equal we cannot use the notion of s-independence that is symmetric. The basic idea of the notion of s-irrelevance is that a class of events **G** is s-irrelevance to class **F** if all events in **G** with the Hausdorff dimension equal to s, are s-irrelevant to all events in **F** with Hausdorff dimension equal to s.

Definition 6. Let Ω=[0,1] and let **F** and **G** be two classes of subsets of Ω. Let us denote $\overline{P}$ and $\underline{P}$ the upper and lower conditional probabilities defined by Hausdorff outer and inner measures.
We say that **G** is s-irrelevant to **F** if for every A∈**F** and B∈ $\boldsymbol{G}^0$ such that $\dim_H$(A)= $\dim_H$(B) we have that B is s-irrelevant to A.

Definition 7. A random variable Y is s-irrelevance to a random variable X if the σ-field generated by Y, is s-irrelevant to the σ-field generated by X.

Example 7. Let X and Y be respectively the indicator functions of the events A=[0,1/2] and B=$\{1/4;1/2\}$; then σ(X)=$\{\Omega, \emptyset, A, A^c\}$ and σ(Y)= $\{\Omega, \emptyset, B, B^c\}$.
Moreover $\overline{P}$ coincides with $\underline{P}$ because the events A and B are measurable with respect to h^1, the Hausdorff measure of order 1 and with respect to h^0, that is the counting measure.
We have that Y is not s-irrelevant to X because

$$P(A|\,B^c) = \frac{h^1(A\cap B^c)}{h^1(B^c)} = \frac{1}{2}\,,\; P(A|\,B) = \frac{h^0(A\cap B)}{h^0(B)} = 1\,.$$

While, recalling that P(A)=P (A| Ω), for every choice of E in σ(X) and H in σ(Y) we have that the factorization property is satisfied and so the random variables are independent with respect to the axiomatic definition.

6 Conclusions

In this paper the notions of s-independence and s-conditional independence with respect to upper and lower conditional probabilities assigned by a class of Hausdorff outer and inner measures are investigated.
The necessity to introduce a new model for coherent upper and lower conditional probabilities arises in the continuous case; in fact coherent upper and lower conditional probabilities cannot always obtained as natural extensions of a regular conditional probability, defined in the approach of Kolmogorov, by the Radon-Nykodym derivative. This happens because when the σ-field of the conditioning events is not countable generated a regular conditional probability cannot satisfy the property P(ω| ω)=1 for every ω belonging to Ω and so its extensions are not separately coherent as required by the definition given by Walley.
Moreover a characterization of coherent conditional probability is given when the set of atoms is finite (see for example [3]). The main result of this paper establishes that the coherence of a probability assessment on a finite family of conditional events can be checked operationally by the satisfiability of a class of sequences of linear systems.
This characterization of coherence can be applied only when the set of atoms is finite (see example 7 of [3]), otherwise you can have that the solution of the first system has all component equal to zero.
So there is the problem to find a tool to define coherent conditional probability and their extensions in the continuous case.
In [10] and [11] coherent upper and lower conditional probabilities are assigned by a class of Hausdorff outer and inner measures. The basic idea of this approach is that *commensurable events* ([6]) with respect to the given (upper) coherent conditional probability, are subsets of Ω with the same Hausdorff dimension. Given a coherent conditional probabilities P* defined on **C** =**F**×$\mathbf{G}^0$, any pair of events A and B of $\mathbf{G}^0$ can be compared as proposed by de Finetti. In fact

$$P^*(A|A\cup B)+P^*(B|A\cup B)\geq 1$$

so the above conditional probabilities cannot be both zero and their ratio can be used to introduce an ordering between A and B. In fact this ratio is finite if either P*(A|A∪B) and P*(B|A∪B) are finite and in this case A and B are called *commensurable*. Otherwise if one of the conditional probability is zero the corresponding event has a probability infinitely less then the other and the two events A and B belong to different *layers* ([7]). We can observe that when conditional probability P* is countable additive there can be only finitely many layers above a given layer, but not so when P* is only finitely additive. Two events A and B of $\mathbf{G}^0$, commensurable with respect to the coherent (upper) conditional probability defined by Theorem 1, are subsets of Ω with the same Hausdorff dimension. The converse is not true, in fact if A is countable and B finite then the two events have Hausdorff dimension equal to 0, but they are not commensurable with respect to the previous conditional probability, since coherence requires that P*(B|A∪B)=0. Two events are commensurable in the sense of de Finetti if and only if they have both finite and positive Hausdorff measure and the same Hausdorff dimension.
With respect to upper and lower conditional probabilities assigned by Hausdorf outer and inner measures the notions of epistemic irrelevance and epistemic independence given by Walley are analised.
As shown by Example 1 when two events have upper probabilities equal to zero we can have that they are logically dependent but epistemicaly independent.
We can observe that this happens for example when Ω is equal to [0,1] and we consider sbubsets with Hausdorff dimension different by that one of Ω.

This problem does not arise if Ω is finite since all non empty subsets of Ω have Hausdorff dimension zero as Ω.
The notions of s-irrelevance and s-independence have been introduced with the aim to assure that logical independence is a necessary condition of stochastic independence.
They are based on the concepts of epistemic irrelevance and epistemic independence proposed by Walley with the further condition that the relative events and their intersection must have the same Hausdorff dimension.
An important different with the notion of independence used in [16], that also assures logical independence as necessary condition of probabilistic independence, is that s-independence does not require any condition about the complements of the events whose independence we are studying.
This is due to the fact that if an event has Hausdorff dimension less than that one of Ω then its complement has Hausdorff dimension equal to Ω.

Moreover s-irrelevance with respect to upper and lower conditional probabilities assigned by Hausdorff outer and inner measures implies strong independence introduced by Levi, that essentially requires that upper and lower conditional probabilities satisfy the factorization property.
S-conditional irrelevance and s-conditional independence are introduced in this paper and it is proved that they imply a generalized factorization property with respect to any conditioning event.
Finally s-irrelevance and s-conditional irrelevance are compared; examples are given to show that the two notions are not related even if the conditioning event is Ω. It is proved that s-irrelevance and s-conditional irrelevance are equivalent when the event A, B, AB and B^c have the same Hausdorff dimension of C= Ω.

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