On eventwise aggregation of coherent lower probabilities

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Abstract

The paper gives sufficient and necessary conditions for eventwise aggregation of various families of lower probabilities, in particular, of coherent lower probabilities, and properties of the corresponding aggregation functions.

Keywords. Coherent lower probabilities, aggregation functions, coherent lower previsions, natural extension.

1 Introduction

The eventwise aggregation appears in decision-making theory, but it is also a good way for constructing fuzzy (non-additive) measures. We should emphases, that well-known representations, including distorted probabilities, decomposable measures, are partial cases of eventwise aggregation for the one-dimensional case [2]. The way of aggregation consists in the following. Let Ω be a measurable space with an algebra $\mathfrak{A}$ and $\mu_1,...,\mu_n$ be fuzzy measures on $\mathfrak{A}$, then, using some monotone function $\varphi:[0,1]^n\to[0,1]$ with $\varphi(0,...,0)=0$, $\varphi(1,...,1)=1$, we get the aggregation $\mu(A)=\varphi(\mu_1(A),...,\mu_n(A))$, $A\in\mathfrak{A}$. It is clear that the above conditions lead to that μ is a fuzzy measure, but in practical issues we should also guarantee that if fuzzy measures $\mu_1,...,\mu_n$ are in a family, say, $\mathcal{M}$, then their aggregation μ is also in $\mathcal{M}$. Such requirement is usually called the condition of consensus or inheritance.

This problem statement was investigated for probability measures in [5, 6], for belief measures in [8], for possibility measures in [3], for decomposable measures in [4]. Some recent results are given in [1], where (generalized) coherent lower probabilities and k-monotone measures are of interest. This paper gives necessary and sufficient conditions for aggregation functions if $\mu_1,...,\mu_n$ are k-monotone, and only sufficient conditions if $\mu_1,...,\mu_n$ are (generalized) coherent lower probabilities, and presents ways for constructing aggregation functions by means of multilinear extension [7]. By the way, it would be important to get also necessary conditions for the pointed cases, giving exact descriptions of such aggregation functions. This problem is solved completely in the article.

The paper has the following structure. First we recall some results obtained in [1], then we investigate families of aggregation functions by the technique, which is similar to well-known constructions like the condition "avoiding sure loss" or "natural extension" in the theory of imprecise probabilities [9], and finally we prove that the sufficient conditions, found in [1] for aggregation functions, are also necessary ones.

2 Preliminaries

Let $\Omega=\{\omega_1,...,\omega_N\}$ be a finite space with the algebra $\mathfrak{A}=2^\Omega$. A set function $\mu:\mathfrak{A}\to[0,1]$ is called a *fuzzy measure* [11] if

1) $\mu(\emptyset)=0$, $\mu(\Omega)=1$;

2) $A,B\in\mathfrak{A}$, $A\subseteq B$ implies $\mu(A)\le\mu(B)$.

Throughout the paper, we denote by $\mathcal{M}_0$ the set of all fuzzy measures on $\mathfrak{A}$; $\mu_1\le\mu_2$, $\mu_1,\mu_2\in\mathcal{M}_0$ if $\mu_1(A)\le\mu_2(A)$ for all $A\in\mathfrak{A}$; let $\mu\in\mathcal{M}_0$ and $\mu(B)\neq 0$ then μ_B is a fuzzy measure, which is expressed by $\mu_B(A)=\frac{\mu(A\cap B)}{\mu(B)}$, $A\in\mathfrak{A}$. The fuzzy measure $\bar\mu$ is called the *dual* of μ if $\bar\mu(A)=1-\mu(\bar A)$, $A\in\mathfrak{A}$. We introduce into consideration the following families of fuzzy measures:

$\mathcal{M}_P$ is the set of all probability measures on $\mathfrak{A}$;

$\mathcal{M}_1=\{\mu\in\mathcal{M}_0\,|\exists P\in\mathcal{M}_P:\mu\le P\}$ is the set of all lower probabilities on $\mathfrak{A}$;

$\mathcal{M}_2=\{\mu\in\mathcal{M}_0\,|\forall B\in\mathfrak{A},\mu(B)\neq 0,\exists P\in\mathcal{M}_P:\mu_B\le P\}$ is the set of all generalized coherent lower probabilities on $\mathfrak{A}$;

$\mathcal{M}_3=\{\mu\in\mathcal{M}_0\,|\forall B\in\mathfrak{A}\ \exists P\in\mathcal{M}_P:\mu\le P,\ \mu(B)=$

$P(B)\}$ is the set of all coherent lower probabilities on $\mathfrak{A}$.

It is easy to check the following embeddings

$$\mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3.$$

Lemma 1 *$\mu \in \mathcal{M}_3$ iff for any $B \in \mathfrak{A}$ there exist $P_1, P_2 \in \mathcal{M}_P$ such that $\mu(AB \cup C\bar{B}) \leq P_1(A)\mu(B) + P_2(C)(1-\mu(B))$ for all $A, C \in \mathfrak{A}$.*

Further we will use pointwise multiplication $\mathbf{xy} = (x_1y_1, ..., x_ny_n)$ of vectors $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n)$ in $[0,1]^n$, and also the inverse operation $\mathbf{x}/\mathbf{y} = (x_1/y_1, ..., x_n/y_n)$ (in the case $x_i = 0$, $y_i = 0$ we determine $x_i/y_i = 0$). We denote $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$, $i = 1, ..., n$.

Let $\mathbf{g} = (g_1, ..., g_n)$ be the n-tuple of $g_1, ..., g_n \in \mathcal{M}_0$ then any *n-aggregation function* $\varphi : [0,1]^n \to [0,1]$ is characterized by $\varphi \circ \mathbf{g} \in \mathcal{M}_0$ and this property holds for any $\mathbf{g} \in \mathcal{M}_0^n$ and any measurable space $(\Omega, \mathfrak{A})$, on which the family $\mathcal{M}_0$ is defined. Notice that each aggregation function φ produces the mapping $\varphi : \mathcal{M}_0^n \to \mathcal{M}_0$ (we use the same notation φ, where $\varphi(\mathbf{g}(A)) = \varphi(g_1(A), ..., g_n(A))$ for all $A \in \mathfrak{A}$). It is clear that an arbitrary function $\varphi : [0,1]^n \to [0,1]$ is not an aggregation function in general, because it may not preserve monotonicity of $\varphi(\mathbf{g})$. The necessary and sufficient condition on aggregation function is given in

Proposition 1 *A function $\varphi : [0,1]^n \to [0,1]$ is an n-aggregation function iff*

1) $\varphi(\mathbf{0}) = 0$, $\varphi(\mathbf{1}) = 1$, where $\mathbf{0} = (0, ..., 0)$, $\mathbf{1} = (1, ..., 1)$.

2) $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \leq \mathbf{y}$ implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

Denote the set of all aggregation functions by $\tilde{\mathcal{M}}_0$. Now we introduce families of aggregation functions, which are closely related to $\mathcal{M}_k$, $k = 1, 2, 3, P$.

$$\tilde{\mathcal{M}}_P = \left\{\varphi \in \tilde{\mathcal{M}}_0 \,|\varphi(\mathbf{x}+\mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in [0,1]^n\right\};$$

$$\tilde{\mathcal{M}}_1 = \left\{\varphi \in \tilde{\mathcal{M}}_0 \,\middle|\, \exists \alpha \in \tilde{\mathcal{M}}_P : \varphi \leq \alpha\right\};$$

$$\tilde{\mathcal{M}}_2 = \left\{\varphi \in \tilde{\mathcal{M}}_0 \,\middle|\, \forall \mathbf{y} \in [0,1]\ \exists \alpha \in \tilde{\mathcal{M}}_P : \varphi(\mathbf{xy}) \leq \alpha(\mathbf{x})\varphi(\mathbf{y}) \text{ for all } \mathbf{x} \in [0,1]^n\right\};$$

$$\tilde{\mathcal{M}}_3 = \left\{\varphi \in \tilde{\mathcal{M}}_0 \,\middle|\, \forall \mathbf{y} \in [0,1]\ \exists \alpha_1, \alpha_2 \in \tilde{\mathcal{M}}_P : \varphi(\mathbf{xy} + \mathbf{z}(\mathbf{1}-\mathbf{y})) \leq \alpha_1(\mathbf{x})\varphi(\mathbf{y}) + \alpha_2(\mathbf{z})(1-\varphi(\mathbf{y})) \text{ for all } \mathbf{x}, \mathbf{z} \in [0,1]^n\right\}.$$

Remark Any $\varphi \in \tilde{\mathcal{M}}_P$ is a linear function [5, 6], i.e. $\varphi(\mathbf{x}) = \sum_{i=1}^n \alpha_i x_i$, where $\alpha_i \geqslant 0$, $i = 1, .., n$, $\sum_{i=1}^n \alpha_i = 1$, $\mathbf{x} = (x_1, ..., x_n)$.

Next propositions show how families $\tilde{\mathcal{M}}_k$ can be used for aggregation and give some properties of them.

Proposition 2 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then*

1) $\varphi : \mathcal{M}_P^n \to \mathcal{M}_P$ iff $\varphi \in \tilde{\mathcal{M}}_P$.

2) $\varphi : \mathcal{M}_k^n \to \mathcal{M}_k$, $k = 1, 2, 3$, if $\varphi \in \tilde{\mathcal{M}}_k$.

Proposition 3 *Let $\mathbf{g} = (g_1, ..., g_n)$ be the n-tuple of $g_1, ..., g_n \in \tilde{\mathcal{M}}_k$, $k = 1, 2, 3$, and $\varphi \in \tilde{\mathcal{M}}_k$ then $\varphi \circ \mathbf{g} \in \tilde{\mathcal{M}}_k$.*

It is easy to check the following embeddings

$$\tilde{\mathcal{M}}_0 \supset \tilde{\mathcal{M}}_1 \supset \tilde{\mathcal{M}}_2 \supset \tilde{\mathcal{M}}_3.$$

The above results become more understandable with the help of the algebra of fuzzy sets. Consider fuzzy subsets of the space $Z = \{1, 2, ..., n\}$. Then by definition a fuzzy subset A of Z is a mapping $A : Z \to [0,1]$. Clearly fuzzy sets generalize ordinary crisp sets. In this terminology, crisp sets are identified with their characteristic functions, i.e. if A is a crisp set then $A(i) = 1$ if $i \in A$, and $A(i) = 0$ otherwise. We introduce the following algebraic operations on fuzzy sets to describe aggregation functions.

1) $C = A \cap B$ (or $C = AB$) if $C(i) = A(i)B(i)$, $i = 1, ..., n$.

2) $\bar{A}$ is the complement of A if $\bar{A}(i) = 1 - A(i)$, $i = 1, ..., n$.

3) $C = A + B$ if $C(i) = A(i) + B(i)$, $i = 1, ..., n$. (This operation is interpreted as union for disjoint crisp sets.)

4) $A \subseteq B$ (or $A \leq B$) if $A(i) \leq B(i)$, $i = 1, ..., n$.

The notion of fuzzy measure on the algebra 2^Z is generalized naturally to the algebra $\tilde{\mathfrak{A}}$ consisting of all fuzzy subsets of Z. By definition [12], a mapping $\varphi : \tilde{\mathfrak{A}} \to [0,1]$ is called a *fuzzy measure* if

1) $\varphi(\emptyset) = 0$, $\varphi(Z) = 1$;

2) $A, B \in \tilde{\mathfrak{A}}$, $A \subseteq B$ implies $\varphi(A) \leq \varphi(B)$.

This enables to generalize all definitions for ordinary fuzzy measures to fuzzy measures on $\tilde{\mathfrak{A}}$, in particular, the fuzzy measure $\bar{\varphi}$ is called the *dual* of φ. if $\bar{\varphi}(A) = 1 - \varphi(\bar{A})$, $A \in \tilde{\mathfrak{A}}$. Let φ a fuzzy measure on $\tilde{\mathfrak{A}}$ and $\varphi(B) \neq 0$ then φ_B is a fuzzy measure, calculated by $\varphi_B(A) = \frac{\varphi(A \cap B)}{\varphi(B)}$, $A \in \tilde{\mathfrak{A}}$.

We will identify fuzzy subsets of Z and vectors in $[0,1]^n$ by $(A(1), ..., A(n)) \in [0,1]^n$, where $A \in \tilde{\mathfrak{A}}$. Then according to Proposition 1 φ is an n-aggregation function if φ is a fuzzy measure on $\tilde{\mathfrak{A}}$. With the help of fuzzy sets, we can describe introduced classes of aggregation functions as

$\varphi \in \tilde{\mathcal{M}}_P$ if $\varphi(A+B) = \varphi(A) + \varphi(B)$ for all $A, B, A+B \in \tilde{\mathfrak{A}}$;

$\varphi \in \tilde{\mathcal{M}}_1$ if there exists a $P \in \tilde{\mathcal{M}}_P$ such that $\varphi \leqslant P$;

$\varphi \in \tilde{\mathcal{M}}_2$ if for any $B \in \tilde{\mathfrak{A}}$ ($\varphi(B) \neq 0$) there exists a $P \in \tilde{\mathcal{M}}_P$ such that $\varphi_B \leqslant P$;

$\varphi \in \tilde{\mathcal{M}}_3$ if for any $B \in \tilde{\mathfrak{A}}$ there exist $P_1, P_2 \in \tilde{\mathcal{M}}_P$ $\varphi(AB + C\bar{B}) \leq P_1(A)\varphi(B) + P_2(C)(1 - \varphi(B))$ for all $A, C \in \tilde{\mathfrak{A}}$.

Let $\tilde{\varphi} \in \tilde{\mathcal{M}}_k$, $k \in \{1, 2, 3, P\}$, and φ is the restriction of $\tilde{\varphi}$ to the algebra 2^Z. It is easily seen that $\varphi \in \mathcal{M}_k$[1]. Such property allows to call introduced classes of aggregation functions with the same name as corresponding families of fuzzy measures, i.e. $\tilde{\varphi}$ is a probability measure for $k = P$, lower probability for $k = 1$, $\tilde{\varphi}$ is a generalized coherent lower probability for $k = 2$, $\tilde{\varphi}$ is a coherent lower probability for $k = 3$.

In [1] it is also investigated the problem how to extend $\varphi \in \mathcal{M}_k$, $k \in \{1, 2, 3, P\}$, on 2^Z to the algebra $\tilde{\mathfrak{A}}$ so that its extension $\tilde{\varphi} \in \tilde{\mathcal{M}}_k$. It turns out, we can do it by multilinear extension [7]. Here we will get other extensions for fuzzy measures from $\mathcal{M}_2$ and $\mathcal{M}_3$ by means of constructions, which are similar to natural extension in the theory of imprecise probabilities.

3 Descriptions of $\mathcal{M}_2$ and $\mathcal{M}_3$ through constructions similar to natural extension

We will use the following notations: $\mathbb{R}$ is the set of all real numbers; $\mathbb{R}_+ = \{x \in |x \geq 0\}$ is the set of all non-negative real numbers; $\mathbb{Q}$ is the set of all rational numbers; $\mathbb{Q}_+ = \{x \in \mathbb{Q} | x \geq 0\}$ is the set of all non-negative rational numbers; $[0,1]_{\mathbb{Q}}$ is the set of all rational numbers in $[0,1]$; $\mathbb{Z}_+$ is the set of all non-negative integers, i.e. $\mathbb{Z}_+ = \{0, 1, 2, ...\}$; $\mathbb{N}$ is the set of all natural numbers, i.e. $\mathbb{N} = \{1, 2, ...\}$.

Further we will use the necessary and sufficient condition of $\varphi \in \tilde{\mathcal{M}}_1$, which is known in the literature [9] as "avoiding sure loss".

Lemma 2 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \in \tilde{\mathcal{M}}_1$ iff for any $\alpha_i \in \mathbb{R}_+$, $\mathbf{x}_i \in [0,1]^n$ with $\sum_i \alpha_i \mathbf{x}_i \leqslant \mathbf{1}$, we have $\sum_i \alpha_i \varphi(\mathbf{x}_i) \leqslant 1$.*

Remark Due to monotonicity of φ, in Lemma 2 we can suppose that $\sum_i \alpha_i \mathbf{x}_i = \mathbf{1}$.

Corollary 1 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \notin \tilde{\mathcal{M}}_1$ iff there exist $\alpha_i \in \mathbb{Q}_+$, $\mathbf{x}_i \in [0,1]^n_{\mathbb{Q}}$ such that $\sum_i \alpha_i \mathbf{x}_i \leqslant \mathbf{1}$ and $\sum_i \alpha_i \varphi(\mathbf{x}_i) > 1$.*

[1] It is natural to keep the same notations for classes of fuzzy measures on 2^Z as for fuzzy measures on 2^Ω.

Proof. Let $\varphi \notin \tilde{\mathcal{M}}_1$ then Lemma 2 implies that there exist $\alpha_i \in \mathbb{R}_+$, $\mathbf{x}_i \in [0,1]^n$ such that $\sum_i \alpha_i \mathbf{x}_i \leqslant \mathbf{1}$ and $\sum_i \alpha_i \varphi(\mathbf{x}_i) > 1$. Let $\sum_i \alpha_i \varphi(\mathbf{x}_i) = \Delta$, $\Delta > 1$. The expression $\sum_i \alpha_i \mathbf{x}_i$ can be considered as continuous function of α_i, $\mathbf{x}_i$. Therefore, it is possible to choose $\beta_i \in \mathbb{Q}_+$, $\beta_i \geqslant \alpha_i$, $\mathbf{y}_i \in [0,1]^n_{\mathbb{Q}}$, $\mathbf{y}_i \geqslant \mathbf{x}_i$, such that $\sum_i \beta_i \mathbf{y}_i \leqslant \delta \mathbf{1}$, where $\delta \in \mathbb{Q}$ and $\delta < \Delta$. Taking $\gamma_i = \frac{\beta_i}{\delta}$, we get $\sum_i \gamma_i \mathbf{y}_i \leqslant \mathbf{1}$, and also $\sum_i \gamma_i \varphi(\mathbf{y}_i) \geqslant \sum_i \gamma_i \varphi(\mathbf{x}_i) = \frac{1}{\delta} \sum_i \beta_i \varphi(\mathbf{x}_i) \geqslant \frac{1}{\delta} \sum_i \alpha_i \varphi(\mathbf{x}_i) = \frac{\Delta}{\delta} > 1$. □

Corollary 2 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \notin \tilde{\mathcal{M}}_1$ iff there exist $\mathbf{x}_i \in [0,1]^n_{\mathbb{Q}}$, $b \in \mathbb{N}$, such that $\sum_i \mathbf{x}_i \leqslant b\mathbf{1}$ and $\sum_i \varphi(\mathbf{x}_i) > b$.*

Proof. We will use notations from Corollary 1. It is clear that numbers α_i can be represented as $\alpha_i = \frac{b_i}{b}$, where $b, b_i \in \mathbb{N}$. Multiplying left and right sides of inequalities from Corollary 1 on b, we get $\sum_i b_i \mathbf{x}_i \leqslant b\mathbf{1}$, $\sum_i b_i \varphi(\mathbf{x}_i) > b$. Notice now that each item $b_i \mathbf{x}_i$ or $b_i \varphi(\mathbf{x}_i)$ can be represented as a finite sum of $\mathbf{x}_i$ or $\varphi(\mathbf{x}_i)$ respectively, whence we get the result required.□

Now we recall some results linked with coherent lower previsions [9]. It is of interest, because they can be considered as aggregation functions. By definition, $\varphi \in \tilde{\mathcal{M}}_0$ is a coherent lower prevision if for any $\mathbf{y} \in [0,1]^n$ there exist a $P \in \tilde{\mathcal{M}}_P$ such that $\varphi \leqslant P$ and $\varphi(\mathbf{y}) = P(\mathbf{y})$.

Proposition 4 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then the following statements are equivalent*

1. *φ is a coherent lower prevision;*
2. *φ satisfies the following conditions:*
   (a) *$\varphi(a\mathbf{x} + c\mathbf{1}) = a\varphi(\mathbf{x}) + c$, $\mathbf{x}, a\mathbf{x} + c\mathbf{1} \in [0,1]^n$, $a \in \mathbb{R}_+$, $c \in \mathbb{R}$;*
   (b) *$\varphi(\mathbf{x} + \mathbf{y}) \geqslant \varphi(\mathbf{x}) + \varphi(\mathbf{y})$, $\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} \in [0,1]^n$;*
   (c) *$\bar{\varphi}(\mathbf{x} + \mathbf{y}) \leqslant \bar{\varphi}(\mathbf{x}) + \bar{\varphi}(\mathbf{y})$, $\mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} \in [0,1]^n$;*
3. *$\varphi(\mathbf{y}) = \sup\{\sum_i \alpha_i \varphi(\mathbf{x}_i) - c | \sum_i \alpha_i \mathbf{x}_i \leqslant \mathbf{y} + c\mathbf{1}$, $\mathbf{x}_i \in [0,1]^n, c, \alpha_i \in \mathbb{R}_+\}$ for all $\mathbf{y} \in [0,1]^n$.*

Lemma 3 *Let φ be a coherent lower prevision then $\varphi \in \tilde{\mathcal{M}}_3$.*

Proof. Let $\varphi(\mathbf{y}) \in (0,1)$. (Other cases, where $\varphi(\mathbf{y}) = 0$ or $\varphi(\mathbf{y}) = 0$, are considered by analogy.) By definition, there exists $P \in \tilde{\mathcal{M}}_P$ such that $\varphi \leqslant P$ and $\varphi(\mathbf{y}) = P(\mathbf{y})$. Introduce into consideration $P_1, P_2 \in \tilde{\mathcal{M}}_P$ defined by $P_1(\mathbf{x}) = \frac{P(\mathbf{xy})}{P(\mathbf{y})}$, $P_2(\mathbf{z}) = \frac{P(\mathbf{z}(\mathbf{1}-\mathbf{y}))}{1-P(\mathbf{y})}$, then $\varphi(\mathbf{xy} + \mathbf{z}(\mathbf{1}-\mathbf{y})) \leqslant P(\mathbf{xy} + \mathbf{z}(\mathbf{1}-\mathbf{y})) = P(\mathbf{xy}) + P(\mathbf{z}(\mathbf{1}-\mathbf{y})) = P_1(\mathbf{x})\varphi(\mathbf{y}) + P_2(\mathbf{z})(1 - \varphi(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{z} \in [0,1]^n$, i.e. $\varphi \in \tilde{\mathcal{M}}_3$.□

Remark It is easy to check that any $\varphi \in \tilde{\mathcal{M}}_3$ satisfies conditions 2(b) and 2(c) from Proposition 4, but does not 2(a) in general.

In the theory of imprecise probabilities there is an important construction, called natural extension. It allows to get a coherent lower prevision $\underline{\varphi}$ from $\varphi \in \tilde{\mathcal{M}}_1$ by

$$\underline{\varphi}(\mathbf{y}) = \sup\left\{ \sum_i \alpha_i\varphi(\mathbf{x}_i) - c \,\middle|\, \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{y} + c\mathbf{1}, \mathbf{x}_i \in [0,1]^n, c, \alpha_i \in \mathbb{R}_+ \right\} \text{ for all } \mathbf{y} \in [0,1]^n.$$

By analogy, we introduce a class of aggregation functions, which consists of generalized coherent lower previsions. By definition, $\varphi \in \tilde{\mathcal{M}}_0$ is a generalized coherent lower prevision if for any $\mathbf{y} \in [0,1]^n$ there exists a linear function $L : [0,1]^n \to \mathbb{R}_+$ with $L(\mathbf{0}) = 0$ such that $\varphi \leqslant L$ and $\varphi(\mathbf{y}) = L(\mathbf{y})$. It should be stressed that $L(\mathbf{1}) \neq 1$ in general.

Proposition 5 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then the following statements are equivalent*

1. *φ is a generalized coherent lower prevision;*
2. *φ satisfies the following conditions:*
   (a) *$\varphi(a\mathbf{x}) = a\varphi(\mathbf{x})$, $\mathbf{x}, a\mathbf{x} \in [0,1]^n$, $a \in \mathbb{R}_+$;*
   (b) *$\varphi(\mathbf{x}+\mathbf{y}) \geqslant \varphi(\mathbf{x}) + \varphi(\mathbf{y})$, $\mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y} \in [0,1]^n$;*
3. *$\varphi(\mathbf{y}) = \sup\{\sum_i \alpha_i\varphi(\mathbf{x}_i) | \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{y}, \alpha_i \in \mathbb{R}_+, \mathbf{x}_i \in [0,1]^n\}$ for all $\mathbf{y} \in [0,1]^n$.*

Proof. Statements 1 and 2 are equivalent by Hahn-Banach's Theorem. It is easy to check the condition

$$\varphi(\mathbf{y}) \geqslant \sup\left\{ \sum_i \alpha_i\varphi(\mathbf{x}_i) \,\middle|\, \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{y}, \mathbf{x}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+ \right\} \text{ for all } \mathbf{y} \in [0,1]^n,$$

is necessary that φ is a generalized coherent lower prevision and sup is always achieved. And finally, it is sufficient to check that the function with the property 3 satisfies conditions 2. □

We introduce the construction like natural extension for generalized coherent previsions by

$$\underline{\varphi}(\mathbf{y}) = \sup\left\{ \sum_i \alpha_i\varphi(\mathbf{x}_i) \,\middle|\, \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{y}, \mathbf{x}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+ \right\} \text{ for all } \mathbf{y} \in [0,1]^n, \text{ and any } \varphi \in \tilde{\mathcal{M}}_1.$$

Then $\underline{\varphi}$ is a generalized coherent lower prevision.

Lemma 4 *Let φ be a generalized coherent lower prevision then $\varphi \in \tilde{\mathcal{M}}_2$.*

Proof. Let $\varphi(\mathbf{y}) \neq 0$. By definition, there exist a linear function $L : [0,1]^n \to \mathbb{R}_+$ with $L(\mathbf{0}) = 0$ such that $\varphi \leqslant L$ and $\varphi(\mathbf{y}) = L(\mathbf{y})$. Introduce into consideration $P \in \tilde{\mathcal{M}}_P$ defined by $P(\mathbf{x}) = \frac{L(\mathbf{xy})}{L(\mathbf{y})}$, then $\varphi(\mathbf{xy}) \leqslant L(\mathbf{xy}) = P(\mathbf{x})\varphi(\mathbf{y})$ for all $\mathbf{x} \in [0,1]^n$, i.e. $\varphi \in \tilde{\mathcal{M}}_3$. □

Remark It is easy to check that any $\varphi \in \tilde{\mathcal{M}}_2$ satisfies condition 2(b) from Proposition 5, but does not 2(a) in general.

Lemma 5 *$\varphi \in \tilde{\mathcal{M}}_2$ iff for any $\mathbf{y} \in [0,1]^n$*

$$\varphi(\mathbf{y}) = \sup\left\{ \sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) \,\middle|\, \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{1}, \mathbf{x}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+ \right\}. \quad (1)$$

Proof. According to the definition $\varphi \in \tilde{\mathcal{M}}_2$ if for any $\mathbf{y} \in [0,1]^n$ ($\varphi(\mathbf{y}) \neq 0$) $\varphi_{\mathbf{y}} \in \tilde{\mathcal{M}}_1$, where $\varphi_{\mathbf{y}}(\mathbf{x}) = \frac{\varphi(\mathbf{xy})}{\varphi(\mathbf{y})}$. Taking this into account, the necessary and sufficient condition from Lemma 2 is written as follows

$$1 \geqslant \sup\left\{ \sum_i \alpha_i \frac{\varphi(\mathbf{x}_i\mathbf{y})}{\varphi(\mathbf{y})} \,\middle|\, \sum_{i=1}^m \alpha_i\mathbf{x}_i \leqslant \mathbf{1}, \mathbf{x}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+ \right\}.$$

Multiplying both sides of the last inequality on $\varphi(\mathbf{y})$, we get

$$\varphi(\mathbf{y}) \geqslant \sup\left\{ \sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) \,\middle|\, \sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{1}, \mathbf{x}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+ \right\}.$$

It is clear that sup is always achieved in the above expression, i.e. the lemma is proved. □

Corollary 3 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \notin \tilde{\mathcal{M}}_2$ iff there exist $\mathbf{y} \in [0,1]^n$, $\mathbf{x}_i \in [0,1]^n_{\mathbb{Q}}$, $\alpha_i \in \mathbb{Q}_+$, such that $\sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{1}$ and $\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) > \varphi(\mathbf{y})$.*

Proof. Let $\varphi \notin \tilde{\mathcal{M}}_2$ then Lemma 3 implies that there exist $\mathbf{y}, \mathbf{x}_i \in [0,1]^n$, $\alpha_i \in \mathbb{R}_+$, such that $\sum_i \alpha_i\mathbf{x}_i \leqslant \mathbf{1}$ and $\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) > \varphi(\mathbf{y})$. Let $\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) = \Delta\varphi(\mathbf{y})$, where $\Delta > 1$. The expression $\sum_i \alpha_i\mathbf{x}_i$ with values in $\mathbb{R}^n_+$ can be considered as a continuous function of $\alpha_i, \mathbf{x}_i$. It enables to choose $\alpha'_i \in \mathbb{Q}_+$, $\alpha'_i \geqslant \alpha_i$, $\mathbf{x}'_i \in [0,1]^n_{\mathbb{Q}}$, $\mathbf{x}'_i \geqslant \mathbf{x}_i$, such that $\sum_i \alpha'_i\mathbf{x}'_i \leqslant \delta\mathbf{1}$, where $\delta \in \mathbb{Q}$ and $\delta < \Delta$. Denoting $\alpha''_i = \frac{\alpha'_i}{\delta}$, we get $\sum_i \alpha''_i\mathbf{x}'_i \leqslant \mathbf{1}$, in addition, $\sum_i \alpha''_i\varphi(\mathbf{x}'_i\mathbf{y}) \geqslant \sum_i \alpha''_i\varphi(\mathbf{x}_i\mathbf{y}) = \frac{1}{\delta}\sum_i \alpha'_i\varphi(\mathbf{x}_i\mathbf{y}) \geqslant \frac{1}{\delta}\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}) = \frac{\Delta\varphi(\mathbf{y})}{\delta} > \varphi(\mathbf{y})$.□

Corollary 4 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \notin \tilde{\mathcal{M}}_2$ iff there exist $\mathbf{y} \in [0,1]^n$, $\mathbf{x}_i \in [0,1]^n_{\mathbb{Q}}$, $d \in \mathbb{N}$ such that $\sum_i \mathbf{x}_i \leqslant d$ and $\sum_i \varphi(\mathbf{x}_i\mathbf{y}) > d\varphi(\mathbf{y})$.*

Proof. We will use notations from Corollary 3. The numbers α_i can be represented as $\alpha_i = \frac{k_i}{d}$, where $d, k_i \in \mathbb{N}$. Multiplying left and right sides of inequalities from Corollary 3 on d, we get $\sum_i k_i \mathbf{x}_i \leqslant d\mathbf{1}$ and $\sum_i k_i \varphi(\mathbf{x}_i \mathbf{y}) > d\varphi(\mathbf{y})$. Notice that each item $k_i \mathbf{x}_i$ or $k_i \varphi(\mathbf{x}_i \mathbf{y})$ can be represented as a finite sum of $\mathbf{x}_i$ or $\varphi(\mathbf{x}_i \mathbf{y})$ respectively, whence we get the result required.□

Proposition 6 *Let $\varphi \in \tilde{\mathcal{M}}_1$ and*

$$\underline{\varphi}(\mathbf{y}) = \sup\left\{\sum_i \alpha_i \varphi(\mathbf{a}_i \mathbf{y}) \,\middle|\, \sum_i \alpha_i \mathbf{a}_i \leqslant \mathbf{1},\ \mathbf{a}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+\right\}, \tag{2}$$

where $y \in [0,1]^n$. Then $\underline{\varphi} \in \tilde{\mathcal{M}}_2$.

Proof. Since under the condition $\varphi \in \tilde{\mathcal{M}}_1$, $\sum_i \alpha_i \varphi(\mathbf{a}_i \mathbf{y}) \leqslant 1$ for $\sum_i \alpha_i \mathbf{a}_i \mathbf{y} \leqslant \mathbf{1}$, i.e. sup always exists in (2), in addition, the function $\underline{\varphi}$ is monotone and $\underline{\varphi}(\mathbf{1}) = 1$, i.e. $\underline{\varphi} \in \tilde{\mathcal{M}}_0$. Consider the function ψ, given by

$$\psi(\mathbf{x}) = \sup\left\{\sum_i \alpha_i \varphi(\mathbf{a}_i \mathbf{y}) \,\middle|\, \sum_i \alpha_i \mathbf{a}_i \leqslant \mathbf{x},\ \mathbf{a}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+\right\}, \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n_+$ $\mathbf{y} \in [0,1]^n$. It easy to check that

1) $\psi(\mathbf{0}) = 0$, $\psi(\mathbf{1}) = \underline{\varphi}(\mathbf{y})$;

2) $\psi(c\mathbf{x}) = c\psi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n_+$, $c \in \mathbb{R}_+$;

3) $\psi(\mathbf{x} + \mathbf{z}) \geqslant \psi(\mathbf{x}) + \psi(\mathbf{z})$, $\mathbf{x}, \mathbf{z} \in \mathbb{R}^n_+$.

With the help of Hahn-Banach's Theorem we argue that there is a linear function $L : \mathbb{R}^n_+ \to \mathbb{R}_+$ ($L(\mathbf{0}) = 0$), such that $\psi \leqslant L$ and $\psi(\mathbf{1}) = L(\mathbf{1})$. On the other hand,

$$\underline{\varphi}(\mathbf{x}\mathbf{y}) = \sup\left\{\sum_i \alpha_i \varphi(\mathbf{a}_i \mathbf{x}\mathbf{y}) \,\middle|\, \sum_i \alpha_i \mathbf{a}_i \mathbf{x} \leqslant \mathbf{x},\ \mathbf{a}_i \in [0,1]^n, \alpha_i \in \mathbb{R}_+\right\},$$

i.e. $\underline{\varphi}(\mathbf{x}\mathbf{y}) \leqslant \psi(\mathbf{x}) \leqslant L(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$, therefore, $\frac{\underline{\varphi}(\mathbf{x}\mathbf{y})}{\underline{\varphi}(\mathbf{y})} \leqslant \frac{L(\mathbf{x})}{L(\mathbf{1})} = P(\mathbf{x})$, where obviously $P \in \tilde{\mathcal{M}}_P$.□

The function $\underline{\varphi}$ can be considered as the natural extension of φ to the class of aggregation functions $\tilde{\mathcal{M}}_2$. Actually, Lemma 5 and Proposition 4 imply that

$$\underline{\varphi}(\mathbf{y}) = \inf\left\{\nu(\mathbf{y}) | \nu \in \tilde{\mathcal{M}}_2,\ \nu \geqslant \varphi\right\},\ \mathbf{y} \in [0,1]^n.$$

Proposition 7 *$\varphi \in \tilde{\mathcal{M}}_3$ iff for any $\mathbf{y} \in [0,1]^n$*

$$\varphi(\mathbf{y}) = \sup\left\{\sum_i \alpha_i \varphi(\mathbf{x}_i \mathbf{y} + \mathbf{z}_i(\mathbf{1} - \mathbf{y})) - c \,\middle|\, \sum_i \alpha_i \mathbf{x}_i \leqslant (c+1)\mathbf{1}, \sum_i \alpha_i \mathbf{z}_i \leqslant c\mathbf{1},\ \mathbf{x}_i, \mathbf{z}_i \in [0,1]^n, \alpha_i, c \in \mathbb{R}_+\right\}. \tag{4}$$

Proof. Necessity. Let $\varphi \in \tilde{\mathcal{M}}_3$ then for any $\mathbf{y} \in [0,1]^n$ one can find $P_1, P_2 \in \tilde{\mathcal{M}}_P$, such that $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y})) \leqslant P_1(\mathbf{x})\varphi(\mathbf{y}) + P_2(\mathbf{z})(1 - \varphi(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{z} \in [0,1]^n$. For the sake of convenience, suppose that domains of the linear functions P_1, P_2 are $\mathbb{R}^n_+$, then for $\sum_i \alpha_i \mathbf{x}_i \leqslant (c+1)\mathbf{1}$, $\sum_i \alpha_i \mathbf{z}_i \leqslant c\mathbf{1}$, $\mathbf{x}_i, \mathbf{z}_i \in [0,1]^n$, $\alpha_i, c \in \mathbb{R}_+$, we get

$$\begin{gathered}\textstyle\sum_i \alpha_i \varphi(\mathbf{x}_i \mathbf{y} + \mathbf{z}_i(\mathbf{1} - \mathbf{y})) - c \leqslant \\ \textstyle\sum_i (P_1(\mathbf{x}_i)\varphi(\mathbf{y}) + P_2(\mathbf{z}_i)(1 - \varphi(\mathbf{y}))) - c = \\ P_1\left(\textstyle\sum_i \mathbf{x}_i\right)\varphi(\mathbf{y}) + P_2\left(\textstyle\sum_i \mathbf{z}_i\right)(1 - \varphi(\mathbf{y})) - c = \\ P_1((c+1)\mathbf{1})\varphi(\mathbf{y}) + P_2(c\mathbf{1})(1 - \varphi(\mathbf{y})) - c = \\ (c+1)\varphi(\mathbf{y}) + c(1 - \varphi(\mathbf{y})) - c = \varphi(\mathbf{y}).\end{gathered}$$

One can see that sup is always achieved in (4).

Sufficiency. Let φ obey (4) then $\varphi \in \tilde{\mathcal{M}}_2$ by Lemma 5. Further, we will consider $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y}))$ as a function of $(\mathbf{x}, \mathbf{z}) \in [0,1]^n \times [0,1]^n$ for a fixed $\mathbf{y} \in [0,1]^n$, also introduce into consideration the auxiliary function

$$\psi(\mathbf{x}, \mathbf{z}) = \sup\left\{\sum_i \alpha_i \varphi(\mathbf{a}_i \mathbf{y} + \mathbf{b}_i(\mathbf{1} - \mathbf{y})) - c \,\middle|\, \sum_i \alpha_i(\mathbf{a}_i, \mathbf{b}_i) \leqslant (\mathbf{x}, \mathbf{z}) + c(\mathbf{1}, \mathbf{1}),\ \mathbf{a}_i, \mathbf{b}_i \in [0,1]^n, \alpha_i, c \in \mathbb{R}_+\right\}. \tag{5}$$

Notice that the last formula is the natural extension of $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y}))$. It exists if there is a linear function $L : [0,1]^n \times [0,1]^n \to [0,1]$ with $L(\mathbf{0}, \mathbf{0}) = 0$ and $L(\mathbf{1}, \mathbf{1}) = 1$, such that $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y})) \leqslant L(\mathbf{x}, \mathbf{z})$ for all $(\mathbf{x}, \mathbf{z}) \in [0,1]^n \times [0,1]^n$. Since $\varphi \in \tilde{\mathcal{M}}_2$, this function may be chosen as $L(\mathbf{x}, \mathbf{z}) = P(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y}))$, where $P \geqslant \varphi$ and $P \in \tilde{\mathcal{M}}_P$. Hence, ψ is a coherent lower prevision. It means that for any $(\mathbf{x}, \mathbf{z}) \in [0,1]^n \times [0,1]^n$ there is a linear function $L : [0,1]^n \times [0,1]^n \to [0,1]$, such that $L \geqslant \psi$ and $L(\mathbf{x}, \mathbf{z}) = \psi(\mathbf{x}, \mathbf{z})$. On the other hand, the right sides of (4) and (5) coincide for $(\mathbf{x}, \mathbf{z}) = (\mathbf{1}, \mathbf{0})$, hence $\varphi(\mathbf{y}) = L(\mathbf{1}, \mathbf{0})$. Taking this into account, we get $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y})) \leqslant \psi(\mathbf{x}, \mathbf{z}) \leqslant L(\mathbf{x}, \mathbf{z}) = L(\mathbf{x}, \mathbf{0}) + L(\mathbf{0}, \mathbf{z})$.

Let $\varphi(\mathbf{y}) \in (0,1)$, consider linear functions $P_1(\mathbf{x}) = \frac{L(\mathbf{x},\mathbf{0})}{L(\mathbf{1},\mathbf{0})}$, $P_2(\mathbf{z}) = \frac{L(\mathbf{0},\mathbf{z})}{L(\mathbf{0},\mathbf{1})}$. It is clear that $P_1, P_2 \in \tilde{\mathcal{M}}_P$ and $\varphi(\mathbf{x}\mathbf{y} + \mathbf{z}(\mathbf{1} - \mathbf{y})) \leqslant P_1(\mathbf{x})\varphi(\mathbf{y}) + P_2(\mathbf{z})(1 - \varphi(\mathbf{y}))$ for all $\mathbf{x}, \mathbf{z} \in [0,1]^n$, i.e. $\varphi \in \tilde{\mathcal{M}}_3$. The cases $\varphi(\mathbf{y}) = 0$, $\varphi(\mathbf{y}) = 1$ are considered by analogy. $P_2 \in \tilde{\mathcal{M}}_P$ is chosen arbitrary for $\varphi(\mathbf{y}) = 0$, $P_1 \in \tilde{\mathcal{M}}_P$ is chosen arbitrary for $\varphi(\mathbf{y}) = 1$.□

Corollary 5 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi \notin \tilde{\mathcal{M}}_3$ iff there exist $\mathbf{y} \in [0,1]^n$, $c, \alpha_i \in \mathbb{Q}_+$, $\mathbf{x}_i, \mathbf{z}_i \in [0,1]^n_{\mathbb{Q}}$, such that $\sum_i \alpha_i \mathbf{x}_i \leqslant (c+1)\mathbf{1}$, $\sum_i \alpha_i \mathbf{z}_i \leqslant c\mathbf{1}$ $\sum_i \alpha_i \varphi(\mathbf{x}_i \mathbf{y} + \mathbf{z}_i(\mathbf{1} - \mathbf{y})) > \varphi(\mathbf{y}) + c$.*

Proof. Let $\varphi \notin \tilde{\mathcal{M}}_3$ then Proposition 5 implies that there exist $\mathbf{y},\mathbf{x}_i,\mathbf{z}_i \in [0,1]^n$, $c,\alpha_i \in \mathbb{R}_+$, such that $\sum_i \alpha_i \mathbf{x}_i \leqslant (c+1)\mathbf{1}$, $\sum_i \alpha_i \mathbf{z}_i \leqslant c\mathbf{1}$ $\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y})) > \varphi(\mathbf{y})+c$. It is clear that c can be taken in $\mathbb{Q}_+$. (If $c\notin\mathbb{Q}_+$ then we can exchange c to any rational q, for which $\varphi(\mathbf{y})+c<q<\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y}))$.) Let $\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y})) = \Delta(\varphi(\mathbf{y})+c)$, where $\Delta>1$. The expression $\sum_i \alpha_i(\mathbf{x}_i,\mathbf{z}_i)$, taking its values in $\mathbb{R}^n_+\times\mathbb{R}^n_+$, can be considered as a continuous function of $\alpha_i,(\mathbf{x}_i,\mathbf{z}_i)$. It enables to choose $\alpha'_i\in\mathbb{Q}_+$, $\alpha'_i\geqslant\alpha_i$, $(\mathbf{x}'_i,\mathbf{z}'_i)\in[0,1]^n_{\mathbb{Q}}\times[0,1]^n_{\mathbb{Q}}$, $(\mathbf{x}'_i,\mathbf{z}'_i)\geqslant(\mathbf{x}_i,\mathbf{z}_i)$, such that $\sum_i\alpha'_i(\mathbf{x}'_i,\mathbf{z}'_i)\leqslant\delta((\mathbf{1},\mathbf{0})+c(\mathbf{1},\mathbf{1}))$, where $\delta\in\mathbb{Q}$ and $\delta<\Delta$. Denoting $\alpha''_i=\frac{\alpha'_i}{\delta}$, we get $\sum_i\alpha''_i(\mathbf{x}'_i,\mathbf{z}'_i)\leqslant(\mathbf{1},\mathbf{0})+c(\mathbf{1},\mathbf{1})$, in addition,

$$\begin{aligned}&\sum_i \alpha''_i\varphi(\mathbf{x}'_i\mathbf{y}+\mathbf{z}'_i(\mathbf{1}-\mathbf{y})) \geqslant\\ &\sum_i \alpha''_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y})) =\\ &\tfrac{1}{\delta}\sum_i \alpha'_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y})) \geqslant\\ &\tfrac{1}{\delta}\sum_i \alpha_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y})) =\\ &\tfrac{\Delta(\varphi(\mathbf{y})+c)}{\delta} > \varphi(\mathbf{y})+c. \ \square\end{aligned}$$

Corollary 6 *Let $\varphi\in\tilde{\mathcal{M}}_0$ then $\varphi\notin\tilde{\mathcal{M}}_3$ if there are $\mathbf{y}\in[0,1]^n$, $\mathbf{x}_i,\mathbf{z}_i\in[0,1]^n_{\mathbb{Q}}$, $b\in\mathbb{Z}_+$, $d\in\mathbb{N}$, such that $\sum_i\mathbf{x}_i\leqslant(b+d)\mathbf{1}$, $\sum_i\mathbf{z}_i\leqslant b\mathbf{1}$ and $\sum_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y}))>d\varphi(\mathbf{y})+b$.*

Proof. We will use notations from Corollary 5. The numbers c, α_i can be represented as $c=\frac{b}{d}$, $\alpha_i=\frac{k_i}{d}$, where $b,k_i\in\mathbb{Z}_+$, $d\in\mathbb{N}$. Multiplying left and right sides of inequalities from Corollary 5 on d, we get $\sum_i k_i\mathbf{x}_i\leqslant(b+d)\mathbf{1}$, $\sum_i k_i\mathbf{z}_i\leqslant b\mathbf{1}$ and $\sum_i k_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y}))>d\varphi(\mathbf{y})+b$. Notice that each item $k_i\mathbf{x}_i$, $k_i\mathbf{z}_i$, or $k_i\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y}))$ can be represented as a finite sum of $\mathbf{x}_i$, $\mathbf{z}_i$, or $\varphi(\mathbf{x}_i\mathbf{y}+\mathbf{z}_i(\mathbf{1}-\mathbf{y}))$, respectively, whence we get the result required.□

Proposition 8 *Let $\varphi\in\tilde{\mathcal{M}}_1$ and*

$$\begin{aligned}\underline{\varphi}(\mathbf{y}) = \ &\sup\Big\{\sum_i\alpha_i\varphi(\mathbf{a}_i\mathbf{y}+\mathbf{b}_i(\mathbf{1}-\mathbf{y}))-c\Big|\\ &\sum_i\alpha_i\mathbf{a}_i\leqslant(c+1)\mathbf{1},\sum_i\alpha_i\mathbf{b}_i\leqslant c\mathbf{1},\\ &\mathbf{a}_i,\mathbf{b}_i\in[0,1]^n,\alpha_i,c\in\mathbb{R}_+\},\end{aligned}\tag{6}$$

where $y\in[0,1]^n$. Then $\underline{\varphi}\in\tilde{\mathcal{M}}_3$.

Proof. Consider the function ψ, calculated by (5) for a fixed $y\in[0,1]^n$. Then $\underline{\varphi}(\mathbf{y})=\psi(\mathbf{1},\mathbf{0})$, and, since ψ is a coherent lower prevision, for any $(\mathbf{x},\mathbf{z})\in[0,1]^n\times[0,1]^n$ one can find a linear function $L:[0,1]^n\times[0,1]^n\to[0,1]$, such that $L\geqslant\psi$ and $L(\mathbf{x},\mathbf{z})=\psi(\mathbf{x},\mathbf{z})$. Thus, for $(\mathbf{x},\mathbf{z})=(\mathbf{1},\mathbf{0})$, as in the above proposition, we imply that there are $P_1,P_2\in\tilde{\mathcal{M}}_P$ with $\psi(\mathbf{x},\mathbf{z})\leqslant P_1(\mathbf{x})\varphi(\mathbf{y})+P_2(\mathbf{z})(1-\underline{\varphi}(\mathbf{y}))$ for all $\mathbf{x},\mathbf{z}\in[0,1]^n$. On the other hand,

$$\begin{aligned}&\underline{\varphi}(\mathbf{xy}+\mathbf{z}(\mathbf{1}-\mathbf{y})) =\\ &\quad\sup\Big\{\sum_i\alpha_i\varphi([\mathbf{a}_i\mathbf{x}+\mathbf{b}_i(1-\mathbf{x})]\,\mathbf{y}+\\ &\quad[\mathbf{a}_i\mathbf{z}+\mathbf{b}_i(1-\mathbf{z})]\,(\mathbf{1}-\mathbf{y}))-c\big|\\ &\quad\sum_i\alpha_i[\mathbf{a}_i\mathbf{x}+\mathbf{b}_i(1-\mathbf{x})]\leqslant\mathbf{x}+c\mathbf{1},\\ &\quad\sum_i\alpha_i[\mathbf{a}_i\mathbf{z}+\mathbf{b}_i(1-\mathbf{z})]\leqslant\mathbf{z}+c\mathbf{1},\\ &\quad\mathbf{a}_i,\mathbf{b}_i\in[0,1]^n,\alpha_i,c\in\mathbb{R}_+\}.\end{aligned}$$

The last expression implies that $\underline{\varphi}(\mathbf{xy}+\mathbf{z}(\mathbf{1}-\mathbf{y}))\leqslant\psi(\mathbf{x},\mathbf{z})$ for all $\mathbf{x},\mathbf{z}\in[0,1]^n$. It means that $\underline{\varphi}(\mathbf{xy}+\mathbf{z}(\mathbf{1}-\mathbf{y}))\leqslant P_1(\mathbf{x})\underline{\varphi}(\mathbf{y})+P_2(\mathbf{z})(1-\underline{\varphi}(\mathbf{y}))$ for all $\mathbf{x},\mathbf{z}\in[0,1]^n$, i.e. $\underline{\varphi}\in\tilde{\mathcal{M}}_3$.□

The function $\underline{\varphi}$ can be considered as the natural extension of φ to the class of aggregation functions $\tilde{\mathcal{M}}_3$. Actually, Propositions 5 and 6 imply that

$$\underline{\varphi}(\mathbf{y})=\inf\left\{\nu(\mathbf{y})|\nu\in\tilde{\mathcal{M}}_3,\ \nu\geqslant\varphi\right\},\ \ \mathbf{y}\in[0,1]^n.$$

In practical applications we can tackle the following problem. Suppose that a function $\varphi_{|\mathfrak{S}}$ is defined on a set $\mathfrak{S}\subseteq[0,1]^n$ and it is required to build the function

$$\underline{\varphi}(\mathbf{y})=\inf\left\{\nu(\mathbf{y})|\nu\in\tilde{\mathcal{M}},\forall\mathbf{x}\in\mathfrak{S}\ \nu(\mathbf{x})\geqslant\varphi_{|\mathfrak{S}}(\mathbf{x})\right\},$$

where $\mathbf{y}\in[0,1]$, $\tilde{\mathcal{M}}$ can be chosen as a set of all generalized coherent previsions, or of all generalized coherent probabilities ($\tilde{\mathcal{M}}=\tilde{\mathcal{M}}_2$), or of all coherent lower probabilities ($\tilde{\mathcal{M}}=\tilde{\mathcal{M}}_3$). It is clear that this problem is soluble if there exists a $P\in\tilde{\mathcal{M}}_P$ with $P(\mathbf{x})\geqslant\varphi_{|\mathfrak{S}}(\mathbf{x})$ for all $\mathbf{x}\in\mathfrak{S}$. Taking in account the resemblance with coherent lower previsions, further such $\underline{\varphi}$ is called the natural extension of $\varphi_{|\mathfrak{S}}$ in $\tilde{\mathcal{M}}$. Without decreasing generality, we can assume that $\mathbf{0},\mathbf{1}\in\mathfrak{S}$ and $\varphi_{|\mathfrak{S}}(\mathbf{0})=0$, $\varphi_{|\mathfrak{S}}(\mathbf{1})=1$. The explicit expression of $\underline{\varphi}$ can be got by using the inner extension of $\varphi_{|\mathfrak{S}}$: $\varphi(\mathbf{y})=\sup\{\varphi_{|\mathfrak{S}}(\mathbf{x})|\mathbf{x}\geqslant\mathbf{y},\ \mathbf{x}\in\mathfrak{S}\}$, and formulas (e.g. (2) and (6)), which give the explicit expression of natural extension in $\tilde{\mathcal{M}}$. It is easy to see that if $\mathfrak{S}$ is finite then the practical calculation of $\underline{\varphi}$ is a linear programming problem. If $\mathfrak{S}$ coincides with the set of all Boolean vectors, i.e. $\mathfrak{S}=\{0,1\}^n$, $\varphi_{|\mathfrak{S}}$ can be perceived as a set function. Next results show that the last case is greatly simplified if $\tilde{\mathcal{M}}=\tilde{\mathcal{M}}_k$, $k=2,3$.

Let $Z=\{1,...,n\}$ and $\mathbf{x}\in[0,1]^n$ then we denote $A_{\mathbf{x}}=\{k\in Z|x_k=1\}$, where $\mathbf{x}=(x_1,...,x_n)$.

Lemma 6 *Let φ be a generalized coherent lower probability on 2^Z then the function $\tilde{\varphi}(\mathbf{x})=\varphi(A_{\mathbf{x}})$, $\mathbf{x}\in[0,1]^n$, is in $\tilde{\mathcal{M}}_2$.*

Proof. To proof the lemma, it is sufficient to find for any $\mathbf{y} \in [0,1]^n$ a probability measure $\tilde{P} \in \tilde{\mathcal{M}}_P$ such that $\tilde{\varphi}(\mathbf{xy}) \leqslant \tilde{P}(\mathbf{x})\tilde{\varphi}(\mathbf{y})$ for all $\mathbf{x},\mathbf{z} \in [0,1]^n$. Because φ is a generalized coherent lower probability on 2^Z, we can take a probability measure P on 2^Z, such that $\varphi(BA_{\mathbf{y}}) \leqslant P(B)\varphi(A_{\mathbf{y}})$ for all $B \in 2^Z$. Further we take $\tilde{P} \in \tilde{\mathcal{M}}_P$ with $\tilde{P}(1_A) = P(A)$ for all $A \in 2^Z$. It is worth to mention that $\tilde{P} \in \tilde{\mathcal{M}}_P$ is defined uniquely and such extension can be obtained by multilinear extension. And finally, we notice that $A_{\mathbf{xy}} = A_{\mathbf{x}}A_{\mathbf{y}}$, $\tilde{\varphi}(\mathbf{y}) = \varphi(A_{\mathbf{y}})$, $\tilde{P}(\mathbf{x}) \geqslant P(A_{\mathbf{x}})$, and $\tilde{\varphi}(\mathbf{xy}) = \varphi(A_{\mathbf{x}}A_{\mathbf{y}}) \leqslant P(A_{\mathbf{x}})\varphi(A_{\mathbf{y}}) \leqslant \tilde{P}(\mathbf{x})\tilde{\varphi}(\mathbf{y})$.□

Lemma 7 *Let φ be a coherent lower probability on 2^Z then the function $\tilde{\varphi}(\mathbf{x}) = \varphi(A_{\mathbf{x}})$, $\mathbf{x} \in [0,1]^n$, is in $\tilde{\mathcal{M}}_3$.*

Proof. To proof the lemma, it is sufficient to find for any $\mathbf{y} \in [0,1]^n$ probability measures $\tilde{P}_1, \tilde{P}_2 \in \tilde{\mathcal{M}}_P$ such that $\tilde{\varphi}(\mathbf{xy} + \mathbf{z}(\mathbf{1}-\mathbf{y})) \leqslant \tilde{P}_1(\mathbf{x})\tilde{\varphi}(\mathbf{y}) + \tilde{P}_2(\mathbf{z})(1 - \tilde{\varphi}(\mathbf{y}))$ for all $\mathbf{x},\mathbf{z} \in [0,1]^n$. Because φ is a coherent lower probability on 2^Z, we can take probability measures P_1, P_2 on 2^Z, such that $\varphi\left(BA_{\mathbf{y}} + C\overline{A_{\mathbf{y}}}\right) \leqslant P_1(B)\varphi(A_{\mathbf{y}}) + P_2(C)(1 - \varphi(A_{\mathbf{y}}))$ for all $B, C \in 2^Z$. Further we take $\tilde{P}_1, \tilde{P}_2 \in \tilde{\mathcal{M}}_P$ with $\tilde{P}_k(1_A) = P_k(A)$, for all $A \in 2^Z$, $k = 1,2$. It is worth to mention that $\tilde{P}_1, \tilde{P}_2 \in \tilde{\mathcal{M}}_P$ are defined uniquely and such extensions can be obtained by multilinear extension. And finally, we notice that $A_{\mathbf{xy}+\mathbf{z}(\mathbf{1}-\mathbf{y})} \subseteq A_{\mathbf{x}}A_{\mathbf{y}} \cup A_{\mathbf{z}}\overline{A_{\mathbf{y}}}$, $\tilde{\varphi}(\mathbf{y}) = \varphi(A_{\mathbf{y}})$, $\tilde{P}_1(\mathbf{x}) \geqslant P_1(A_{\mathbf{x}})$,$\tilde{P}_2(\mathbf{z}) \geqslant P_2(A_{\mathbf{z}})$, and $\tilde{\varphi}(\mathbf{xy}+\mathbf{z}(\mathbf{1}-\mathbf{y})) \leqslant \varphi\left(A_{\mathbf{x}}A_{\mathbf{y}} \cup A_{\mathbf{z}}\overline{A_{\mathbf{y}}}\right) \leqslant P_1(A_{\mathbf{x}})\varphi(A_{\mathbf{y}}) + P_2(A_{\mathbf{z}})(1 - \varphi(A_{\mathbf{y}})) \leqslant \tilde{P}_1(\mathbf{x})\tilde{\varphi}(\mathbf{y}) + \tilde{P}_2(\mathbf{z})(1 - \tilde{\varphi}(\mathbf{y}))$.□

Example 1 Consider the fuzzy measure φ on 2^Z, where $Z = \{1,2,3\}$, defined by $\varphi(\{1,2,3\}) = 1$, $\varphi(\{1,2\}) = 2/3$, $\varphi(\{2,3\}) = 2/3$, and φ is supposed to be equal to zero on other sets in 2^Z. It is easily checked that φ is a generalized coherent lower probability, but not a coherent lower probability on 2^Z. We can build a coherent lower probability, taking a natural extension of φ. As result, we get a coherent lower probability ψ such that $\psi(\{2\}) = 1/3$ and ψ is equal to φ on other sets in 2^Z. It is easy to see that ψ is totally monotone or a belief measure on 2^Z. Consider now various aggregation functions, generated by φ. We denote by $\wedge$ the minimum operation, and $\mathbf{x} = (x_1, x_2, x_3)$.

$\tilde{\varphi}(\mathbf{x}) = \varphi(A_{\mathbf{x}})$ is the natural extension of φ in $\tilde{\mathcal{M}}_2$ by Lemma 6.

$\tilde{\psi}(\mathbf{x}) = \psi(A_{\mathbf{x}})$ is in $\tilde{\mathcal{M}}_3$ by Lemma 7. ($\tilde{\psi}$ is the the natural extension of φ in the set of coherent lower probabilities $\tilde{\mathcal{M}}_3$.)

$\tilde{\varphi}_1(\mathbf{x}) = \frac{1}{3}x_2 + \frac{1}{3}(x_1 \wedge x_2) + \frac{1}{3}(x_2 \wedge x_3)$ is the natural extension of φ in the set of coherent lower previsions (i.e. in usual sense) and $\tilde{\varphi}_1 \in \tilde{\mathcal{M}}_3$;

$\tilde{\varphi}_2(\mathbf{x}) = \tilde{\varphi}_1(\mathbf{x}) \wedge \left(\frac{2}{3}x_1 + \frac{2}{3}x_3\right)$ is the natural extension of φ in the set of generalized coherent previsions and $\tilde{\varphi}_2 \in \tilde{\mathcal{M}}_2$;

$\tilde{\varphi}_3(\mathbf{x}) = \frac{2}{3}(x_1x_2) + \frac{2}{3}(x_2x_3) - \frac{1}{3}(x_1x_2x_3)$ is the multilinear extension[2] of φ and $\tilde{\varphi}_3 \in \tilde{\mathcal{M}}_2$;

$\tilde{\varphi}_4(\mathbf{x}) = \frac{1}{3}x_2 + \frac{1}{3}(x_1x_2) + \frac{1}{3}(x_2x_3)$ is the multilinear extension of ψ and $\tilde{\varphi}_4 \in \tilde{\mathcal{M}}_3$.

Next results give the simple expressions of natural extension in $\tilde{\mathcal{M}}_k$, $k = 2,3$, if we consider the one-dimensional case of aggregation functions.

Lemma 8 *Let we consider 1-aggregation functions and $\tilde{\mathcal{M}}$ be a non-empty subset of $\tilde{\mathcal{M}}_k$, $k = 1,2$. Then the function $\varphi(y) = \sup\left\{\nu(y) | \nu \in \tilde{\mathcal{M}}\right\}$, $y \in [0,1]$ is in $\tilde{\mathcal{M}}_k$.*

Proof. Let $k = 2$ then any $\nu \in \tilde{\mathcal{M}}$ is characterized by $\nu(xy) \leqslant x\nu(y)$ for all $x,y \in [0,1]$. It is clear that also $\varphi(xy) \leqslant x\varphi(y)$ for all $x,y \in [0,1]$, i.e. $\varphi \in \tilde{\mathcal{M}}_2$.

Let $k = 3$ then any $\nu \in \tilde{\mathcal{M}}$ is bi-elastic [1, 10], i.e. $\nu(xy) \leqslant x\nu(y)$ and $\nu(y+z) \leqslant (1-z)\nu(y) + z$ for all $x,y,z \in [0,1]$. It is clear φ is also bi-elastic, i.e. $\varphi \in \tilde{\mathcal{M}}_3$. □

Lemma 9 *Let $\mathfrak{S}$ be an non-empty subset of $(0,1)$, $\varphi : \mathfrak{S} \to [0,1]$ and $\varphi(x) \leqslant x$ for all $x \in \mathfrak{S}$, then the natural extension $\underline{\varphi}_k$ of φ in $\tilde{\mathcal{M}}_k$ is given by*

$$\underline{\varphi}_2(y) = \sup_{x \in \mathfrak{S}} \nu^{(1)}_{|x,\varphi}(y),\ y \in [0,1],$$

$$\underline{\varphi}_3(y) = \sup_{x \in \mathfrak{S}} \max\left\{\nu^{(1)}_{|x,\varphi}(y), \nu^{(2)}_{|x,\varphi}(y)\right\},\ y \in [0,1],$$

where

$$\nu^{(1)}_{|x,\varphi}(y) = \begin{cases} 0, & 0 \leqslant y < x, \\ \varphi(x)y/x, & x \leqslant y < 1, \\ 1, & y = 1, \end{cases}$$

$$\nu^{(2)}_{|x,\varphi}(y) = \begin{cases} 1 - \dfrac{(1-\varphi(x))(1-y)}{(1-x)}, & 0 \leqslant y < x, \\ \varphi(x), & x \leqslant y \leqslant 1. \end{cases}$$

Proof. According to the definition

$$\underline{\varphi}_k(y) = \inf\left\{\nu(y) | \nu \in \tilde{\mathcal{M}}_k, \forall x \in \mathfrak{S}\ \varphi(x) \leqslant \nu(x)\right\}.$$

[2]The multilinear extension of φ can be calculated with the help of the Möbius transform m of φ: $m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|}\varphi(B)$, $A \in 2^Z$, by the formula $\tilde{\varphi}_3(\mathbf{x}) = \sum_{B \subseteq Z} m(B) \prod_{i \in B} x_i$, $\mathbf{x} \in [0,1]^n$.

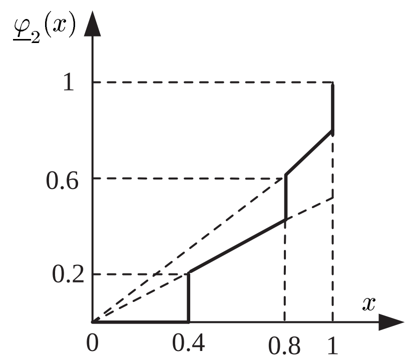

Figure 1: The natural extension of φ in $\tilde{\mathcal{M}}_2$.

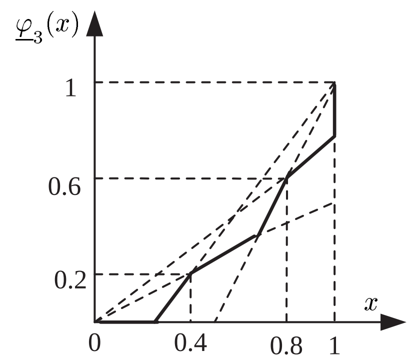

Figure 2: The natural extension of φ in $\tilde{\mathcal{M}}_3$.

It is easily checked that

$$\nu^{(1)}_{|x,\varphi} = \inf\left\{\nu(y)|\nu \in \tilde{\mathcal{M}}_2, \varphi(x) \leqslant \nu(x)\right\},$$

$$\max\left\{\nu^{(1)}_{|x,\varphi}(y), \nu^{(2)}_{|x,\varphi}(y)\right\} = \inf\left\{\nu(y)|\nu \in \tilde{\mathcal{M}}_3 : \varphi(x) \leqslant \nu(x)\right\},$$

where $x \in \mathfrak{S}$ and $y \in [0,1]$, i.e. $\nu^{(1)}_{|x,\varphi}(y) \leqslant \underline{\varphi}_2(y)$ and $\max\left\{\nu^{(1)}_{|x,\varphi}(y), \nu^{(2)}_{|x,\varphi}(y)\right\} \leqslant \underline{\varphi}_3(y)$ for all $y \in [0,1]$, i.e. $\underline{\varphi}_2(y) \geqslant \sup_{x\in\mathfrak{S}} \nu^{(1)}_{|x,\varphi}(y)$ and $\underline{\varphi}_3(y) \geqslant \sup_{x\in\mathfrak{S}} \max\left\{\nu^{(1)}_{|x,\varphi}(y), \nu^{(2)}_{|x,\varphi}(y)\right\}$. On the other hand, according to Lemma 8, the left side of the first inequality is a function in $\tilde{\mathcal{M}}_2$, and the left side of the second inequality is a function in $\tilde{\mathcal{M}}_3$. This proves the lemma.□

Example 2 Consider the function $\varphi : \mathfrak{S} \to [0,1]$, where $\mathfrak{S} = \{0.4, 0.8\}$, $\varphi(0.4) = 0.2$, $\varphi(0.8) = 0.6$. Then we construct functions $\underline{\varphi}_k$, $k = 2,3$, which are depicted in Figures 1, 2.

4 Necessary conditions of $\varphi : \mathcal{M}^n_k \to \mathcal{M}_k$, $k = 1,2,3$

Next propositions are based on the following lemma.

Lemma 10 *Let $\sum_{m=1}^M x_m = b$, where $x_m \in [0,1]_{\mathbb{Q}}$, $m = 1,...,M$, $b \in \mathbb{Z}_+$. Then it is possible to construct a finite probability space $(\Omega, \mathfrak{A}, P)$, in which some family of sets $\{A_m\}_{m=1}^M \subseteq \mathfrak{A}$ can be chosen as follows: $P(A_m) = x_m$, $m = 1,...,M$, $\sum_{m=1}^M 1_{A_m} = b1_\Omega$.*

Proof. Let $b \in \mathbb{N}$. (The case $b = 0$ is obvious.) Rational numbers x_m can be represented as $x_m = \frac{k_m}{K}$, where $K \in \mathbb{N}$, $k_m \in \{0,1,...,K\}$, $m = 1,...,M$. Choose probability space with $\Omega = \{1,...,K\}$, $\mathfrak{A} = 2^\Omega$, $P(\{m\}) = \frac{1}{K}$ for any $m \in \Omega$. Then the solving problem can be formulated as follows. Given a set of pairs of indices $I = \{\langle i,j\rangle \,| i \in \Omega, j \in \{1,...,b\}\}$. It is required to find a partition $\{B_1, B_2, ..., B_M\}$ of I ($\bigcup_{m=1}^M B_m = I$, $B_l \cap B_m = \emptyset$ for $l \neq m$), such that

1) $|B_m| = k_m$, $m = 1,...,M$;

2) $\langle i,j_1\rangle \in B_m$, $\langle i,j_2\rangle \in B_m \Rightarrow j_1 = j_2$.

Then we choose $A_m = \{i|\,\langle i,j\rangle \in B_m\}$.

Assign natural numbers to elements of I in dictionary order, i.e. suppose that the number of $\langle i,j\rangle$ is $N(\langle i,j\rangle) = i + K(j-1)$. It is obvious that $N(\langle i_1,j_1\rangle) > N(\langle i_2,j_2\rangle)$ for $j_1 > j_2$, and $N(\langle i_1,j\rangle) > N(\langle i_2,j\rangle)$ for $i_1 > i_2$. Using this order, we fill up the sets $B_1 = \{\langle i,j\rangle \,|\, N(\langle i,j\rangle) \in \{1,...,k_1\}\}$,..., $B_m = \{\langle i,j\rangle \,|\, N(\langle i,j\rangle) \in \{k_1 + ... + k_{m-1} + 1, ..., k_1 + ... + k_m\}$,..., $B_M = \{\langle i,j\rangle \,| N(\langle i,j\rangle) \in \{Kb - k_m + 1, ..., Kb\}$. It is easy to see that sets B_m satisfy the conditions required, namely, $\bigcup_{m=1}^M B_m = I$, $B_l \cap B_m = \emptyset$ for $l \neq m$, $|B_m| = k_m$, $m = 1,...,M$. In addition, if $\langle i_1,j_1\rangle, \langle i_2,j_2\rangle \in B_m$, then $|N(\langle i_1,j_1\rangle) - N(\langle i_2,j_2\rangle)| \leqslant k_m - 1$. Suppose now, the condition 2) is not fulfilled, i.e. there are $\langle i,j_1\rangle, \langle i,j_2\rangle \in B_m$ with $j_1 > j_2$, then $N(\langle i,j_1\rangle) - N(\langle i,j_2\rangle) = K(j_1 - j_2) \geqslant k_m$, which contradicts to the inequality obtained.□

Consider also the corollary of the above lemma, which are used for proving properties of aggregation functions of fuzzy measures from $\mathcal{M}_2$ and $\mathcal{M}_3$.

Corollary 7 *Let $y \in [0,1]$, $\sum_{m=1}^M x_m = b + d$, $\sum_{m=1}^M z_m = b$, where $x_m, z_m \in [0,1]_{\mathbb{Q}}$, $m = 1,...,M$, and $b \in \mathbb{Z}_+$, $d \in \mathbb{N}$. Then one can construct a finite probability space $(\Omega, \mathfrak{A}, P)$, in which sets $B \in \mathfrak{A}$, $\{A_m\}_{m=1}^M, \{C_m\}_{m=1}^M \subseteq \mathfrak{A}$ have the following properties: $P(B) = y$, $P(A_m) = x_m y$, $P(C_m) = z_m(1-y)$,*

$m=1,...,M$, $\sum_{m=1}^{M} 1_{A_m} = (b+d)1_B$, $\sum_{m=1}^{M} 1_{C_m} = b1_{\bar{B}}$.

Proof. By Lemma 10, we can construct two probability spaces $(\Omega', \mathfrak{A}', P')$, $\mathfrak{A}' = 2^{\Omega'}$, $(\Omega'', \mathfrak{A}'', P'')$, $\mathfrak{A}'' = 2^{\Omega''}$, in which sets $\{A_m\}_{m=1}^{M} \subseteq \mathfrak{A}'$, $\{C_m\}_{m=1}^{M} \subseteq \mathfrak{A}''$ have the following properties: $P'(A_m) = x_m$, $P''(C_m) = z_m$, $m=1,...,M$, $\sum_{m=1}^{M} 1_{A_m} = (b+d)1_{\Omega'}$, $\sum_{m=1}^{M} 1_{C_m} = b1_{\Omega''}$. In addition, we can suppose that $\Omega' \cap \Omega'' = \emptyset$. Further we construct the required probability space $(\Omega, \mathfrak{A}, P)$, where $\mathfrak{A} = 2^{\Omega' \cup \Omega''}$ and $P(A) = yP'(A \cap \Omega') + (1-y)P''(A \cap \Omega'')$, $A \in \mathfrak{A}$. It is obvious that the probability measure P is well suited for $B = \Omega'$.□

Consider some slight modification in the aggregation of fuzzy measures. Let $\Omega_1, ..., \Omega_n$ be finite, pairwise disjoint nonempty sets, $\mathfrak{A}_i$ be the power set of Ω_i, and also $\Omega = \bigcup_{i=1}^{n} \Omega_i$, $\mathfrak{A} = 2^{\Omega}$. Then for fuzzy measures $\mu_i : \mathfrak{A}_i \to [0,1]$ the rule of aggregation, based on an aggregation function $\varphi : [0,1]^n \to [0,1]$, is defined by $\mu(A) = \varphi(\mu_1(A \cap \Omega_1), ..., \mu_n(A \cap \Omega_n))$, where $A \in \mathfrak{A}$. We should stress that the last rule is a particular case of the considered eventwise aggregation, because set functions $\mu_1(A \cap \Omega_1), ..., \mu_n(A \cap \Omega_n)$ of $A \in \mathfrak{A}$ can be considered as extensions of $\mu_1, ..., \mu_n$ to the algebra $\mathfrak{A}$. By the way, such rule of aggregation is more convenient in further investigations, since values of $\mu_1(A \cap \Omega_1), ..., \mu_n(A \cap \Omega_n)$ can be chosen independently.

Proposition 9 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi : \mathcal{M}_1 \to \mathcal{M}_1$ iff $\varphi \in \tilde{\mathcal{M}}_1$.*

Proof. We should prove only necessity (see Proposition 2). Let $\varphi \notin \tilde{\mathcal{M}}_1$ then by Corollary 2 there exist $\mathbf{x}_i \in [0,1]_{\mathbb{Q}}^n$, $b \in \mathbb{N}$, such that $\sum_{i=1}^{M} \mathbf{x}_i \leqslant b\mathbf{1}$ and $\sum_{i=1}^{M} \varphi(\mathbf{x}_i) > b$. Let $\mathbf{x}_i = (x_{1i}, ..., x_{ni})$, then by Lemma 10 we can construct probability spaces $(\Omega_k, \mathfrak{A}_k, P_k)$, $\mathfrak{A}_k = 2^{\Omega_k}$, $k = 1, ..., n$, in which sets $\{A_{km}\}_{m=1}^{M} \subseteq \mathfrak{A}_k$ are linked with the sequence $(x_{k1}, ..., x_{kM})$ by $P(A_{km}) = x_{km}$, $m = 1, ..., M$, $\sum_{m=1}^{M} 1_{A_{km}} = b(1_{\Omega_k})$. Let $\Omega = \bigcup_{k=1}^{n} \Omega_k$, $\mathfrak{A} = 2^{\Omega}$ (we can suppose $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$), and consider the aggregation of probability measures P_k, $k = 1, ..., n$, produced by $\mu(A) = \varphi(P_1(A \cap \Omega_1), ..., P_n(A \cap \Omega_n))$, $A \in \mathfrak{A}$. Let us check that $\mu \in \mathcal{M}_1$. If it is true, one can find $P \in \mathcal{M}_P$ with $P \geqslant \mu$. Since $\sum_{k=1}^{n} \sum_{m=1}^{M} 1_{A_{km}} = b(1_{\Omega})$, we get

$$\sum_{m=1}^{M} P\left(\bigcup\nolimits_{k=1}^{n} A_{km}\right) = \sum_{i=1}^{b} P(\Omega) = \sum_{i=1}^{b} 1 = b.$$

On the other hand,

$$\sum_{m=1}^{M} P\left(\bigcup\nolimits_{k=1}^{n} A_{km}\right) \geqslant \sum_{m=1}^{M} \mu\left(\bigcup\nolimits_{k=1}^{n} A_{km}\right) =$$
$$\sum_{m=1}^{M} \varphi(P_1(A_{1m}), ..., P_n(A_{nm})) = \sum_{i=1}^{M} \varphi(\mathbf{x}_i) > b.$$

The contradiction obtained implies that $\mu \notin \mathcal{M}_1$.□

Proposition 10 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi : \mathcal{M}_2 \to \mathcal{M}_2$ iff $\varphi \in \tilde{\mathcal{M}}_2$.*

Proof. We should prove only necessity (see Proposition 2). Let $\varphi \notin \tilde{\mathcal{M}}_2$ then by Corollary 4 there exist $\mathbf{y} \in [0,1]^n$, $\mathbf{x}_i \in [0,1]_{\mathbb{Q}}^n$, $d \in \mathbb{N}$, such that $\sum_{i=1}^{M} \mathbf{x}_i = d$ and $\sum_{i=1}^{M} \varphi(\mathbf{x}_i \mathbf{y}) > d\varphi(\mathbf{y})$. Let $\mathbf{y}_i = (y_1, ..., y_n)$, $\mathbf{x}_i = (x_{1i}, ..., x_{ni})$, then by Corollary 7, for any $k \in \{1, ..., n\}$ we can construct a finite probability space $(\Omega_k, \mathfrak{A}_k, P_k)$, $\mathfrak{A}_k = 2^{\Omega_k}$, in which there are sets $B_k, A_{km} \in \mathfrak{A}_k$, $m = 1, ..., M$, with $P_k(B_k) = y_k$, $P_k(A_{km}) = y_k x_{km}$, $m = 1, ..., M$, $\sum_{m=1}^{M} 1_{A_{km}} = d1_{B_k}$. Let $\Omega = \bigcup_{k=1}^{n} \Omega_k$, $\mathfrak{A} = 2^{\Omega}$ (we can suppose that $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$, and the aggregation of probability measures P_k is produced by $\mu(A) = \varphi(P_1(A \cap \Omega_1), ..., P_n(A \cap \Omega_n))$, $A \in \mathfrak{A}$. Let us check that $\mu \in \mathcal{M}_2$. If it is true then there exists a $P^{(1)} \in \mathcal{M}_P$, such that $\mu(AB) \leqslant P^{(1)}(A)\mu(B)$ for all $A \in \mathfrak{A}$ and $B = \bigcup_{k=1}^{n} B_k$. It is clear that $\mu(B) = \varphi(\mathbf{y})$, and also it is easy to see that $P^{(1)}(B) = 1$. Denote $A_m = \bigcup_{k=1}^{n} A_{km}$ then $\sum_{m=1}^{M} 1_{A_m} = d1_B$, and we get

$$\sum_{m=1}^{M} \mu(A_m B) \leqslant \mu(B) \sum_{m=1}^{M} P^{(1)}(A_m) =$$
$$\varphi(\mathbf{y}) \sum_{m=1}^{d} P^{(1)}(B) = d\varphi(\mathbf{y}).$$

On the other hand,

$$\sum_{m=1}^{M} \mu(A_m B) = \sum_{m=1}^{M} \varphi(P_1(A_{1m} B_1), ..., P_n(A_{nm} B_n))$$
$$= \sum_{m=1}^{M} \varphi(\mathbf{x}_m \mathbf{y}) > d\varphi(\mathbf{y}).$$

The contradiction obtained implies that $\mu \notin \mathcal{M}_2$.□

Proposition 11 *Let $\varphi \in \tilde{\mathcal{M}}_0$ then $\varphi : \mathcal{M}_3 \to \mathcal{M}_3$ iff $\varphi \in \tilde{\mathcal{M}}_3$.*

Proof. We should prove only necessity (see Proposition 2). Let $\varphi \notin \tilde{\mathcal{M}}_3$ then by Corollary 6 there exist $\mathbf{y} \in [0,1]^n$, $\mathbf{x}_i, \mathbf{z}_i \in [0,1]_{\mathbb{Q}}^n$, $b \in \mathbb{Z}_+$, $d \in \mathbb{N}$, such that $\sum_{i=1}^{M} \mathbf{x}_i = (b+d)\mathbf{1}$, $\sum_{i=1}^{M} \mathbf{z}_i = b\mathbf{1}$ and $\sum_{i=1}^{M} \varphi(\mathbf{x}_i \mathbf{y} + \mathbf{z}_i(\mathbf{1} - \mathbf{y})) > d\varphi(\mathbf{y}) + b$. Let $\mathbf{y}_i = (y_1, ..., y_n)$, $\mathbf{x}_i = (x_{1i}, ..., x_{ni})$, $\mathbf{z}_i = (z_{1i}, ..., z_{ni})$, then by Corollary 7, for any $k \in \{1, ..., n\}$ we can construct a finite probability space $(\Omega_k, \mathfrak{A}_k, P_k)$, $\mathfrak{A}_k = 2^{\Omega_k}$, in which there are sets $B_k, C_{km}, A_{km} \in \mathfrak{A}_k$, $m = 1, ..., M$, with $P_k(B_k) = y_k$, $P_k(A_{km}) = y_k x_{km}$, $P_k(C_{km}) = (1-y_k)z_{km}$, $m = 1, ..., M$, $\sum_{m=1}^{M} 1_{A_{km}} =$

$(b+d)1_{B_k}$, $\sum_{m=1}^M 1_{C_{km}} = b1_{\bar{B}_k}$. Let $\Omega = \bigcup_{k=1}^n \Omega_k$, $\mathfrak{A} = 2^\Omega$ (we can suppose that $\Omega_k \cap \Omega_l = \emptyset$ for $k \neq l$), and the aggregation of probability measures P_k is produced by $\mu(A) = \varphi\left(P_1\left(A \cap \Omega_1\right), ..., P_n\left(A \cap \Omega_n\right)\right)$, $A \in \mathfrak{A}$. Let us check that $\mu \in \mathcal{M}_3$. If it is true then there exist $P^{(1)}, P^{(2)} \in \mathcal{M}_P$, such that $\mu(AB \cup C\bar{B}) \leqslant P^{(1)}(A)\mu(B) + P^{(2)}(C)(1-\mu(B))$ for all $A, C \in \mathfrak{A}$ and $B = \bigcup_{k=1}^n B_k$. It is clear that $\mu(B) = \varphi(\mathbf{y})$, also it is possible to suppose that $P^{(1)}(B) = 1$ and $P^{(2)}(\bar{B}) = 1$. Denote $A_m = \bigcup_{k=1}^n A_{km}$, $C_m = \bigcup_{k=1}^n C_{km}$, then $\sum_{m=1}^M 1_{A_m} = (b+d)1_B$, $\sum_{m=1}^M 1_{C_m} = b1_{\bar{B}}$, and

$$\sum_{m=1}^M \mu(A_m B \cup C_m \bar{B}) \leqslant \mu(B) \sum_{m=1}^M P^{(1)}(A_m) + (1-\mu(B))$$

$$\sum_{m=1}^M P^{(2)}(C_m) = \varphi(\mathbf{y}) \sum_{m=1}^{b+d} P^{(1)}(B) + (1-\varphi(\mathbf{y}))$$

$$\sum_{m=1}^b P^{(2)}(\bar{B}) = (b+d)\varphi(\mathbf{y}) + (1-\varphi(\mathbf{y}))b = d\varphi(\mathbf{y}) + b.$$

On the other hand, $\sum_{m=1}^M \mu(A_m B \cup C_m \bar{B}) =$

$$\sum_{m=1}^M \varphi\left(P_1\left(A_{1m}B_1 \cup C_{1m}\bar{B}_1\right), ..., P_n\left(A_{nm}B_n \cup C_{nm}\bar{B}_n\right)\right)$$

$$= \sum_{i=1}^M \varphi\left(\mathbf{x}_i \mathbf{y} + \mathbf{z}_i(\mathbf{1}-\mathbf{y})\right) > d\varphi\left(\mathbf{y}\right) + b.$$

The contradiction obtained implies that $\mu \notin \mathcal{M}_3$.□

5 Conclusion

This paper and [1] give us the full description of aggregation functions (or functionals), which can be used for eventwise aggregation of fuzzy measures from various families of lower probabilities, including lower probabilities, generalized coherent lower probabilities, coherent lower probabilities, k-monotone and belief measures. This investigation covers also the one-dimensional case, giving distorted probabilities (or distorted fuzzy measures). For example, these results allow to argue that each distortion function for coherent lower probabilities has to be chosen from the family of bi-elastic functions [10]. One can see that the aggregation problem leads to introducing various functionals and constructions, which are similar to the known ones in the theory of imprecise probabilities like "coherent lower previsions" or "natural extension".

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References

[1] A.G. Bronevich. On the closure of families of fuzzy measures under eventwise aggregations. *Fuzzy sets and systems*, 153: 45–70, 2005.

[2] A. Chateauneuf. Decomposable capacities, distorted probabilities and concave capacities. *Mathematical Social Sciences*, 31:19–37, 1996.

[3] D. Dubois, H. Prade. Aggregation of Possibility Measures, in: J. Kacprzyk, M. Fedrizzi (eds.) *Multiperson Decision Making Using Fuzzy Sets and Possibility Theory.* Kluwer Academic Publishers, Dordrecht, 55–63, 1990.

[4] D. Dubois, J.C. Fodor, H. Prade, M. Roubens. Aggregation of decomposable measures with application to utility theory. *Theory and Decision*, 41: 59-95, 1996.

[5] K. Lehrer, C.G. Wagner. *Rational consensus in Science and Society.* D. Reidel Publ. Co., Boston, 1981.

[6] K. McConway. Marginalization and linear opinion pools. *J. Amer. Statictical Assoc.*, 76: 410–414, 1981.

[7] G. Owen. Multilinear extensions of games, in: A.E. Roth (ed.) *The Shapley Value. Essays in Honor of Lloyd S. Shapley.* Cambridge University Press, Cambridge, UK, 139–151, 1988.

[8] C.G. Wagner. Consensus for belief functions and related uncertainty measures. *Theory and decision*, 26: 295–304, 1989.

[9] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.

[10] A. Wallner, Bi-elastic Neighbourhood models. *Proc. of the 3-th international symposium on imprecise probabilities and their applications.* Lugano, Switzerland, 591–605, 2003.

[11] Z. Wang, G.J. Klir. *Fuzzy measure theory.* Plenum, New-York, 1992.

[12] Q. Zhong. On fuzzy measure and fuzzy integral on fuzzy set. *Fuzzy sets and Systems*, 37: 77–92, 1990.