A Granular Semantics for Fuzzy Measures and its Application to Climate Change Scenarios

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Abstract

A granular based semantics for fuzzy measures is introduced in which the measure of a set of propositions approximates the probability of the disjunction of these propositions. This approximation is derived from known probabilities across a granular partition of the set of possible worlds. This interpretation is then extended to allow for the case where there is uncertainty regarding the meanings of propositions. Such a semantics is motivated by, and provides some justification for, the use of fuzzy measures to quantify the uncertainty associated with climate emissions scenarios. The use of socio-economic scenarios in climate models is discussed in the context of a possible worlds model and an example is given of the use of fuzzy measures across scenarios to aggregate global mean temperature predictions.

Keywords. Fuzzy measures, operational semantics, uncertain models, emission scenarios

1 Introduction

The notion of fuzzy measure as proposed by Sugeno [12] has been the subject of much theoretical investigation (see [13] for an overview) and has been applied to range of problems in artificial intelligence, with special focus on the problem of aggregation. Despite this interest it remains somewhat unclear how fuzzy measures should be interpreted within an operational setting. This exasperates the practical difficulties of eliciting numerical values from experts (see [2] for a discussion) and makes any uncertainty modelling based on fuzzy measures difficult to to analyse. Formally, a fuzzy measure is defined as a (possibly) non-additive measure satisfying monotonicity with respect to the subsethood relation as follows:

Definition 1 *Fuzzy Measure [12]*
Let X be a finite universe then a function $\mu : 2^X \rightarrow [0,1]$ is a fuzzy measure if it satisfies the following axioms:

- $\mu(X) = 1$, $\mu(\emptyset) = 0$
- *If* $S \subseteq T \subseteq X$ *then* $\mu(S) \leq \mu(T)$

This is a rather general definition and includes a wide range of measures such as, probability measures, Dempster-Shafer belief and Plausibility measures, and Possibility measures. The interpretations of fuzzy measures that have been proposed within the literature fall mainly into two categories. One view taken in [13] and [5] is that the universe X consists of certain indicators of the quality of an individual or object and for $S \subseteq X$, $\mu(S)$ is a measure of the collective importance of the set of indicators S. The second view (as discussed in [5]) proposes that fuzzy measures quantify uncertainty in the same way as probability or Dempster-Shafer measures. It is this second view with which we shall be concerned in this paper. A possible third interpretation of fuzzy measures is provided by [11] in which the power set of the universe X corresponds to the set of possible outcomes of an experiment and $\mu(S)$ is regarded as the fraction of the total available resources consumed if the result of the experiment is S. However, while interesting this semantics would seem more closely related to the importance, rather than the uncertainty interpretation of fuzzy measures.

In the following we outline a semantics for fuzzy measures that utilizes a notion of information granularity similar to that proposed by Pawlak [10] as the basis of rough set theory. In this interpretation the set of possible worlds Ω is partitioned into granules corresponding to minimal subsets for which probability values can be either estimated or elicited. We then suggest using the probabilities of the granules to provide an estimate for the probability of other subsets of Ω as defined by declarative propositions about the world. This estimate is then shown to be a fuzzy measure.

In our case the motivation for investigating semantics for fuzzy measures is drawn from the specific problem of identifying an uncertainty measure to evaluate emissions scenarios in climate change models. Models of future climate change require, amongst other inputs, time series of greenhouse gas emissions. These are derived from scenarios based on linguistic narratives of potential socio-economic futures [8]. As such they are far from precise constructs.

The analysis of uncertainty in climate research has raised fundamental questions about the most appropriate way to represent uncertainties and communicate them to decision-makers. Of central importance is the debate concerning whether climate change predictions should be expressed in probabilistic terms. On the one hand it is argued that probabilities are essential to make rational decisions under conditions of uncertainty. On the other hand, certain aspects of climate uncertainty, in particular emission scenarios because of their imprecise and overlapping nature, do not lend themselves to quantification by a purely additive measure of uncertainty. In the sequel we investigate the use of fuzzy measures to capture the uncertainty associated with emission scenarios and consider if such a model can be justified within the proposed granular semantics.

2 Semantics for Fuzzy Measures

Let $G = \{A_1, \ldots, A_n\}$ and $D = \{B_1, \ldots, B_m\}$ be two distinct sets of propositions about the world. Also let Ω denote the set of all conceivable world descriptions. We then define

$$\mathcal{A}_i = \{w \in \Omega : w \models A_i\} \text{ for } i = 1, \ldots, n$$
$$\mathcal{B}_i = \{w \in \Omega : w \models B_i\} \text{ for } i = 1, \ldots, m$$

Hence, $\mathcal{A}_i$ and $\mathcal{B}_j$ corresponds to the models of propositions A_i and B_j respectively for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. It is also assumed that $\forall i\ \mathcal{A}_i \neq \emptyset$, $\forall i, j : i \neq j\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ and that $\bigcup_{i=1}^{n} \mathcal{A}_i = \Omega$. In other words, $\{\mathcal{A}_i : i = 1, \ldots, n\}$ is a partition of Ω. Intuitively this means that G is a complete description of reality based on only limited granularity. For example, if $x : \Omega \to \mathbb{R}$ is a variable representing some numerical measurement of a world w then a possible description of granularity 2 is $G = \{A_1, A_2\}$ where:

$$A_1 = x > \delta \text{ and } A_2 = x \leq \delta \text{ for } \delta \in \mathbb{R}$$

In this case

$$\mathcal{A}_1 = \{w \in \Omega : x(w) > \delta\} \text{ and}$$
$$\mathcal{A}_2 = \{w \in \Omega : x(w) \leq \delta\}$$

On the other hand, D corresponds to a set of propositions about the world in which we are interested, but which are not necessarily disjoint (i.e. $\mathcal{B}_i \cap \mathcal{B}_j \neq \emptyset$).

Now suppose that we do not have enough information to specify a probability distribution on the complete set of possible worlds Ω but we are able to elicit probability values of the granular propositions $A_i : i = 1, \ldots, n$. Let P be a probability measure on 2^G with corresponding probability distribution p on G. Can we use this probabilistic information regarding G to estimate belief values for propositions in D? We propose a measure μ on 2^D as follows:

Definition 2 *Initially we define*

$$\forall S \subseteq D,\ \textit{let } \mathcal{W}(S) = \bigcup_{B_j \in S} \mathcal{B}_j$$

this being the set of worlds consistent with a least one of the statements in S being true. Then $\forall S \subseteq D, \forall \alpha \in [0, 1]$ let

$$\forall S \subseteq D,\ \mu_\alpha(S) = P(S_\alpha) = \sum_{A_i \in S_\alpha} p(A_i)$$
$$\textit{where } S_\alpha = \left\{ A_i : \frac{|\mathcal{A}_i \cap \mathcal{W}(S)|}{|\mathcal{A}_i|} \geq \alpha \right\}$$

Intuitively then according to Definition 2 $\mu_\alpha(S)$ provides an approximation to the probability of compound proposition $\bigvee_{B_i \in S} B_i$ based on the probability measure P defined on the granular propositions in G. Specifically, this estimate is obtained by first identifying those granular propositions A_i for which their associated models $\mathcal{A}_i$ overlaps $\mathcal{W}(S)$, the model of $\bigvee_{B_i \in S} B_i$, to a degree greater than or equal to α. $\mu_\alpha(S)$ is then evaluated as the sum of the probabilities of these highly overlapping granular propositions. For example, in Figure 1 the grid cells correspond to $\mathcal{A}_i : i = 1, \ldots, n$ and the circles to $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$. In this case, the dark grey cells are those $\mathcal{A}_i$ which completely overlap with (i.e are contained within) $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and the light grey cells are those $\mathcal{A}_i$ that overlap with this set to some degree $\alpha > 0$. The parameter α can be interpreted as quantifying the level of caution employed when estimating probabilities. Effectively the higher the value of α the more likely it is that $\mu_\alpha(S)$ is an underestimate of the probability of $\bigvee_{B_i \in S} B_i$ whereas the lower the value of α the more likely it is that $\mu_\alpha(S)$ is an overestimate of this probability.

The following theorem shows that provided $\mathcal{B}_i : i = 1, \ldots, m$ covers the set of possible worlds Ω then μ_α is a fuzzy measure.

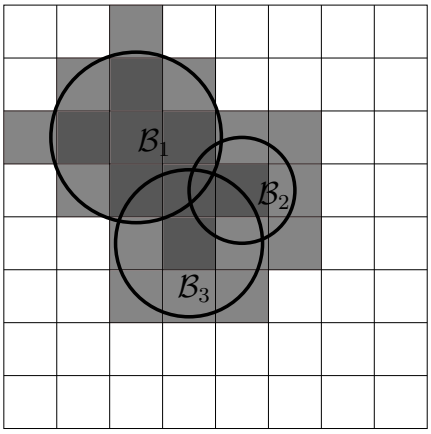

Figure 1: Granules, $\mathcal{A}_i$ consistent with $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. The dark grey cells correspond to those $\mathcal{A}_i : \forall\alpha\ \mathcal{A}_i \in S_\alpha$. The light grey cells correspond to those $\mathcal{A}_i : \exists\alpha > 0,\ \mathcal{A}_i \in S_\alpha$

Theorem 3

$\forall\alpha \in (0,1]$ μ_α *is a fuzzy measure on* 2^D *provided* $\bigcup_{i=1}^{m} \mathcal{B}_i = \Omega$.

Proof

$\mu_\alpha(D) = P(D_\alpha)$ *and*
$D_\alpha = \{A_i : \frac{|\mathcal{A}_i \cap \mathcal{W}(D)|}{|\mathcal{A}_i|} \geq \alpha\} =$
$\{A_i : \frac{|\mathcal{A}_i \cap \Omega|}{|\mathcal{A}_i|} \geq \alpha\} = G$
therefore $\mu_\alpha(D) = P(G) = 1$

Now

$\emptyset_\alpha = \{A_i : \frac{|\mathcal{A}_i \cap \emptyset|}{|\mathcal{A}_i|} \geq \alpha\} = \emptyset$ *since* $\alpha > 0$
and therefore $\mu_\alpha(\emptyset) = P(\emptyset) = 0$

$\forall S_1, S_2 \subseteq D,\ (S_1 \cup S_2)_\alpha =$
$\{\mathcal{A}_i : \frac{|\mathcal{A}_i \cap \mathcal{W}(S_1 \cup S_2)|}{|\mathcal{A}_i|} \geq \alpha\}$

Now

$\forall \mathcal{A}_i \in G,\ |\mathcal{A}_i \cap \mathcal{W}(S_1 \cup S_2)|$
$\geq \max(|\mathcal{A}_i \cap \mathcal{W}(S_1)|, |\mathcal{A}_i \cap \mathcal{W}(S_2)|)$

and therefore

$(S_1 \cup S_2)_\alpha \supseteq (S_1)_\alpha$ *and* $(S_1 \cup S_2)_\alpha \supseteq (S_2)_\alpha$
so that
$P((S_1 \cup S_2)_\alpha) \geq \max(P((S_1)_\alpha), P((S_2)_\alpha))$
which implies that
$\mu_\alpha(S_1 \cup S_2) \geq \max(\mu_\alpha(S_1), \mu_\alpha(S_2))$ $\square$

We can define upper and lower bounds on μ_α as follows:

Upper and Lower Measures

Note that, trivially, $\forall\epsilon \in (0,1]$

$$\bigcup_{\alpha \geq \epsilon} S_\alpha = S_\epsilon$$

Lower Bound:

$\forall S \subseteq D,\ \underline{S} = S_1 = \{A_i : \mathcal{A}_i \subseteq \mathcal{W}(S)\}$
$\underline{\mu}(S) = P(\underline{S})$

Upper Bound:

$\forall S \subseteq D,\ \overline{S} = \bigcup_{\alpha > 0} S_\alpha = \{A_i : \mathcal{A}_i \cap \mathcal{W}(S) \neq \emptyset\}$
$\overline{\mu}(S) = P(\overline{S})$

Trivially

$\forall S \subseteq D,\ \forall\alpha \in (0,1],\ \underline{\mu}(S) \leq \mu_\alpha(S) \leq \overline{\mu}(S)$

Notice that there is a clear link here to Pawlak's theory of rough sets [10] in that the pair $(\underline{S}, \overline{S})$ is a rough set approximation to $\mathcal{W}(S)$. There is also a relationship with Ziarko's variable precision rough set model [15] in that S_α can also be expressed in terms of the generalised inclusion relation, proposed in [15], which allows for a margin of error ϵ in judgements about subsethood, so that:

$$X \overset{\epsilon}{\subseteq} Y \text{ iff } \frac{|X \cap Y|}{|X|} \geq 1 - \epsilon$$

In this notation

$$S_\alpha = \left\{A_i : \mathcal{A}_i \overset{1-\alpha}{\subseteq} \mathcal{W}(S)\right\}$$

The following theorem shows that the upper measure $\overline{\mu}$ can never be super-additive.

Theorem 4

$\forall S_1, S_2 \subseteq D,\ \overline{\mu}(S_1 \cup S_2) \leq \overline{\mu}(S_1) + \overline{\mu}(S_2)$

Proof

$\{A_i : \mathcal{A}_i \cap (\mathcal{W}(S_1 \cup S_2)) \neq \emptyset\} =$
$\{A_i : \mathcal{A}_i \cap (\mathcal{W}(S_1)) \neq \emptyset\}$
$\cup\{A_i : \mathcal{A}_i \cap (\mathcal{W}(S_2)) \neq \emptyset\}$
$\Rightarrow \overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$
$\Rightarrow P\left(\overline{S_1 \cup S_2}\right) \leq P\left(\overline{S_1}\right) + P\left(\overline{S_2}\right)$ $\square$

The following example illustrates both sub-additive and super-additive behaviour from the fuzzy measure μ_α as the parameter α varies within the range $(0, 1]$.

Example 5 *Let $\Omega = \{w_1, \ldots, w_{10}\}$ and suppose $G = \{A_1, \ldots, A_5\}$ and $D = \{B_1, \ldots, B_4\}$ are defined such that:*

$\mathcal{A}_1 = \{w_1, w_2\},\ \mathcal{A}_2 = \{w_3, w_4, \},$
$\mathcal{A}_3 = \{w_5\},\ \mathcal{A}_4 = \{w_6, w_7, w_8\},$
$\mathcal{A}_5 = \{w_9, w_{10}\}$
$\mathcal{B}_1 = \{w_1, w_2, w_4, w_6\}, \mathcal{B}_2 = \{w_1, w_2, w_3, w_5, w_9, w_{10}\},$
$\mathcal{B}_3 = \{w_7, w_9, w_{10}\}, \mathcal{B}_4 = \{w_8, w_{10}\}$

Let

$P(A_1) = 0.1,\ P(A_2) = 0.35,\ P(A_3) = 0.05,$
$P(A_4) = 0.4,\ P(A_5) = 0.1$

Let $\alpha = 0.5$ then

$\{B_3\}_{0.5} = \{A_5\}$ *and* $\{B_4\}_{0.5} = \{A_5\}$*because*
$\dfrac{|\mathcal{A}_4 \cap \mathcal{B}_3|}{|\mathcal{A}_4|} = \dfrac{|\{w_7\}|}{|\{w_6, w_7, w_8\}|} = \dfrac{1}{3} < \dfrac{1}{2}$ *and*
$\dfrac{|\mathcal{A}_5 \cap \mathcal{B}_3|}{|\mathcal{A}_5|} = \dfrac{|\{w_9, w_{10}\}|}{|\{w_9, w_{10}\}|} = 1 > \dfrac{1}{2}$ *and*
$\dfrac{|\mathcal{A}_4 \cap \mathcal{B}_4|}{|\mathcal{A}_4|} = \dfrac{|\{w_8\}|}{|\{w_6, w_7, w_8\}|} = \dfrac{1}{3} < \dfrac{1}{2}$ *and*
$\dfrac{|\mathcal{A}_5 \cap \mathcal{B}_4|}{|\mathcal{A}_5|} = \dfrac{|\{, w_{10}\}|}{|\{w_9, w_{10}\}|} = \dfrac{1}{2} \geq \dfrac{1}{2}$ *and*
$\dfrac{|\mathcal{A}_i \cap \mathcal{B}_3|}{|\mathcal{A}_i|} = \dfrac{|\mathcal{A}_i \cap \mathcal{B}_4|}{|\mathcal{A}_i|} = 0$ *for* $i = 1, \ldots, 3$

Therefore

$\mu_{0.5}(\{B_3\}) = 0.1$ *and* $\mu_{0.5}(\{B_4\}) = 0.1$
$\{B_3, B_4\}_{0.5} = \{A_4, A_5\}$ *because*
$\dfrac{|\mathcal{A}_4 \cap (\mathcal{B}_3 \cup \mathcal{B}_4)|}{|\mathcal{A}_4|} = \dfrac{|\{w_7, w_8\}|}{|\{w_6, w_7, w_8\}|} = \dfrac{2}{3} > \dfrac{1}{2}$
and $\dfrac{|\mathcal{A}_5 \cap (\mathcal{B}_3 \cup \mathcal{B}_5)|}{|\mathcal{A}_5|} = \dfrac{|\{w_9, w_{10}\}|}{|\{w_9, w_{10}\}|} = 1 > \dfrac{1}{2}$
Therefore, $\mu_{0.5}(\{B_3, B_4\}) = 0.1 + 0.4 = 0.5 >$
$\mu_{0.5}(\{B_3\}) + \mu_{0.5}(\{B_4\}) = 0.2$

Also

$\overline{\{B_3\}} = \{A_4, A_5\}$ *and* $\overline{\{B_4\}} = \{A_4, A_5\}$
therefore
$\overline{\mu}(\{B_3\}) = \overline{\mu}(\{B_4\}) = p(A_4) + p(A_5) = 0.5$
furthermore
$\overline{\{B_3, B_4\}} = \{A_4, A_5\} \Rightarrow\ \overline{\mu}(\{B_3, B_4\}) = 0.5$
$< \overline{\mu}(\{B_3\}) + \overline{\mu}(\{B_4\})$

Finally

$\underline{\{B_3\}} = \{A_5\}$ *and* $\underline{\{B_4\}} = \emptyset$ *therefore*
$\underline{\mu}(\{B_3\}) = 0.1$ *and* $\underline{\mu}(\{B_4\}) = 0$
also $\underline{\{B_3, B_4\}} = \{A_5\}$ *therefore*
$\underline{\mu}(\{B_3, B_4\}) = 0.1 = \underline{\mu}(\{B_3\}) + \underline{\mu}(\{B_4\})$

Figure 2 shows how the additivity of $\mu_\alpha(\{B_3, B_4\})$ varies with α. For high values of α this measure is additive, there is an intermediate range of α values for which it is super-additive and for low values we have sub-additivity.

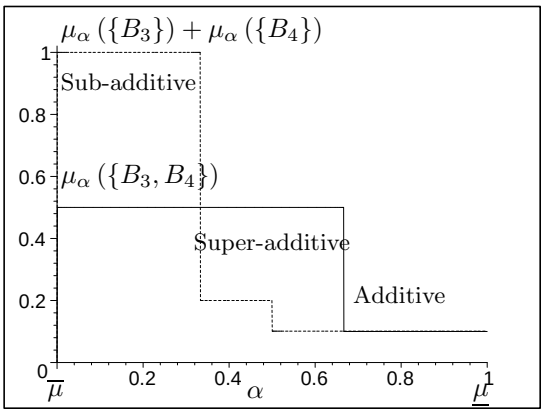

Figure 2: Regions of additivity, sub-additivity and super-additivity of μ_α as α varies. The black line represents the value of $\mu_\alpha(\{B_3, B_4\})$ while the dashed line represents the associated additive value $\mu_\alpha(\{B_3\}) + \mu_\alpha(\{B_4\})$.

3 Fuzzy Measures of Fuzzy Propositions

For many problems the meanings of the propositions under consideration (in this case D) may not be precisely defined. For socio-economic scenarios, in particular, there is no clear consensus as to their exact definitions. Indeed, the fact that they are largely based on linguistic descriptions would suggest a certain variability of definition across a range of experts arising from the natural variability in the use of language.

In this section we propose an extension to the granularity semantics for fuzzy measures that incorporates a range of opinions regarding the models for the propositions in D. Specifically, we define a notion of uncertain models, as derived from a finite set of experts V, in terms of random sets into the power set of Ω.

Definition 6 *Uncertain Models*
Let V be a finite set of experts then we define $\mathcal{B}_i : V \to 2^{\Omega}$ for $i = 1, \ldots, m$ to be a sequence of random sets into 2^{Ω} such that: $\forall v \in V,\ \bigcup_{i=1}^{m} \mathcal{B}_i(v) = \Omega$

Now suppose we have a measurement variable $x : \Omega \to \mathbb{R}$ quantifying a specific property of a given state of the world. For example, in climate modelling x might correspond to the fossil CO_2 emmissions for a particular year. In the case where our knowledge of the state of the world is restricted to some subset of Ω then this naturally defines an interval of values for x as follows:

Definition 7 *Let $x : \Omega \to \mathbb{R}$ then define*

$\overline{x} : 2^{\Omega} \to \mathbb{R}$ *such that* $\forall T \subseteq \Omega$,
$\overline{x}(T) = \max\{w \in T : x(w)\}$
and
$\underline{x} : 2^{\Omega} \to \mathbb{R}$ *such that* $\forall T \subseteq \Omega$,
$\underline{x}(T) = \min\{w \in T : x(w)\}$

Definition 7 assumes a precise constraint on the state of the world of the form $w \in T$ for $T \subseteq \Omega$. However, in the case that our knowledge corresponds to a proposition B_i with an uncertain model as in definition 6, then x can only be known relative to a mass assignment on possible intervals as follows: Let ρ be a probability distribution on V then we can define a mass assignment for random set $\mathcal{B}_i$ according to:

$$\forall T \subseteq \Omega,\ m_i(T) = \rho(\{v \in V : \mathcal{B}_i(v) = T\})$$

This interpretation of mass assignments is an extension of the voting model proposed by Gaines [4] and subsequently developed by Baldwin [1].

Let

$$\mathcal{I}_x = \{[\underline{x}(T), \overline{x}(T)] : T \subseteq 2^{\Omega}\}$$

Then we can define an associated mass assignment on intervals of $\mathbb{R}$ containing x as follows:

$$\forall I \in \mathcal{I}_x,\ m_i^x(I) = \sum_{T \subseteq \Omega : [\underline{x}(T), \overline{x}(T)] = I} m_i(T)$$

Now assuming that we interpret fuzzy membership functions as single point coverage functions of random sets [6] then from the above we can determine a membership function for variable x in scenario B_i as follows:

$$\forall y \in \mathbb{R},\ \pi_i^x(y) = \sum_{I \in \mathcal{I}_x : y \in I} m_i^x(I)$$

Furthermore, in the case that $\mathcal{B}_i$ is a consonant (nested) random set then we can derive m_i^x directly from π_i^x according to the standard algorithm for possibility measures. (see Klir [9])

We now extend Definition 2 to the case where the models for propositions in D are given by random sets based on expert opinion.

Definition 8 *Initially we define*

$$\forall S \subseteq D,\ \forall v \in V,\ \text{let } \mathcal{W}_v(S) = \bigcup_{B_j \in S} \mathcal{B}_j(v)$$

Then $\forall S \subseteq D,\ \forall v \in V,\ \forall \alpha \in (0, 1]$,

$$\text{let } S_{\alpha,v} = \left\{ A_i : \frac{|\mathcal{A}_i \cap \mathcal{W}_v(S)|}{|\mathcal{A}_i|} \geq \alpha \right\}$$

From this we define

$$\forall S \subseteq D,\ \mu_{\alpha}(S) = \sum_{v \in V} \rho(v)\, \mu_{\alpha,v}(S)$$

where $\forall v \in V,\ \text{let } \mu_{\alpha,v}(S) = P(S_{\alpha,v})$

Theorem 9

$\forall \alpha \in (0, 1],\ \mu_{\alpha}$, *as given in Definition 8, is a fuzzy measure on 2^D provided that*

$$\forall v \in V,\ \bigcup_{j=1}^{m} \mathcal{B}_j(v) = \Omega$$

Proof *By Theorem 3 it follows that $\forall \alpha \in (0, 1]$ and $\forall v \in V$, $\mu_{\alpha,v}$ is a fuzzy measure. Therefore, since μ_{α} is a linear combination of such measures with positive coefficients that sum to 1 it follows that μ_{α} is also a fuzzy measure.*

Example 10 *Let $\Omega = \{w_1, \ldots w_{10}\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $D = \{B_1, \ldots B_4\}$ are defined as given in the table in Figure 3.*

Let $x(w_i) = i^2 : i = 1, \ldots, 10$ and let
$\rho(v_1) = 0.1,\ \rho(v_2) = 0.1,$
$\rho(v_3) = 0.4,\ \rho(v_4) = 0.25,\ \rho(v_5) = 0.15$
In this case $m_1(\{w_1, w_2, w_4, w_6\}) = 0.1,$
$m_1(\{w_1, w_2, w_4\}) = 0.1, m_1(\{w_1, w_2\}) = 0.65$
$m_1(\{w_1\}) = 0.15$ *from which we can evaluate*
$m_1^x([1, 36]) = 0.1, m_1^x([1, 16]),$
$m_1^x([1, 4]) = 0.65, m_1^x([1, 1]) = 0.15$

expert	$\mathcal{B}_1(v)$	$\mathcal{B}_2(v)$	$\mathcal{B}_3(v)$	$\mathcal{B}_4(v)$
v_1	$\{w_1, w_2, w_4, w_6\}$	$\{w_3, w_9\}$	$\{w_7, w_5, w_8, w_9, w_{10}\}$	$\{w_{10}\}$
v_2	$\{w_1, w_2, w_4\}$	$\{w_3, w_5, w_6, w_9\}$	$\{w_7, w_9\}$	$\{w_8, w_{10}\}$
v_3	$\{w_1, w_2\}$	$\{w_3, w_4, w_5, w_6, w_9\}$	$\{w_7\}$	$\{w_8, w_9, w_{10}\}$
v_4	$\{w_1, w_2\}$	$\{w_2, w_3, w_4, w_5, w_6, w_9\}$	$\{w_7\}$	$\{w_8, w_{10}\}$
v_5	$\{w_1\}$	$\{w_1, w_2, w_3, w_4, w_5, w_6, w_9\}$	$\{w_7, w_9\}$	$\{w_8, w_{10}\}$

Figure 3: Table showing random set definitions of $\mathcal{B}_1, \ldots, \mathcal{B}_4$

The resulting possibility distribution is then given by (see Figure 4):

$$\pi_1^x = \begin{cases} 1 & : \quad x = 1 \\ 0.85 & : \quad x \in (1, 4] \\ 0.2 & : \quad x \in (4, 16] \\ 0.1 & : \quad x \in (16, 36] \\ 0 & : \quad x > 36 \end{cases}$$

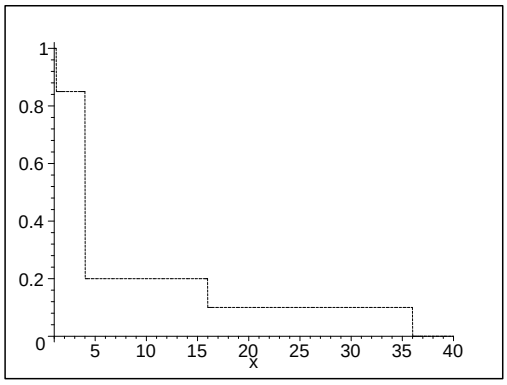

Figure 4: Possibility distribution π_1^x on x

4 Climate Emissions Scenarios

The IPCC special report on emissions scenarios (SRES) [8] identifies a two dimension space within which socio-economic scenarios can be located (see Figure 5) and the UK government Foresight Futures document [3] adopts a similar conceptual framework. The x-axis in Figure 5 corresponds to social values ranging from the individualistic to those associated with a strong community. The y-axis relates to systems of government and their decision making processes. Here the range is from 'autonomy' where power remains largely at the national level to 'interdependent' where power is shared across a range of institutions and between national governments (E.g. as in the EU). Conventional development is seen as lying mostly within the quadrant where the social system emphasises individual values but where we have increasing economic and governmental dependence. Linguistic descriptions of these scenarios are used to derive assumptions upon which quantified modelling of, for example, the global economy and greenhouse gas emissions, is based.

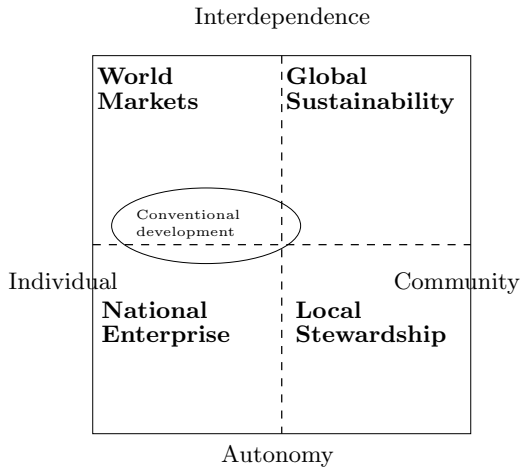

Figure 5: A conceptual space for climate scenarios [3]. The x-axis corresponds to social values and the y-axis to systems of governance.

The points in the space identified in Figure 5 can conceptually be thought of as possible states of the world and scenarios as potentially overlapping regions identified by a set of linguistic descriptions D. Similarly, the granular propositions G would identify a coarser partition of the space at which probability values could realistically be elicited. In practice, of course, the space D occupied by scenarios is likely to be highly multi-dimensional, taking into account a wide range of social, economic and political factors. Indeed the SRES states that this is indeed the case [8]. None the less, the idea that scenarios identify a set of possible worlds within such a space seems intuitively appealing and may provide a conceptual framework to aid in the elicitation of measures of uncertainty.

Emissions scenarios are typically clustered into families where the members of each family are based on roughly equivalent socio-economic descriptions. In

this paper we focus on four families of scenarios: A1, A2, B1 and B2. Summary descriptions of these families as given in [8] are as follows:

- The **A1 scenario family** describe a future world of very rapid economic growth, global population that peaks in mid-century and declines thereafter, and the rapid introduction of new and more efficient technologies. Major underlying themes are convergence among regions, capacity building, and increased cultural and social interactions, with a substantial reduction in regional differences in per capita income.

- The **A2 scenario family** describe a very heterogeneous world. The underlying theme is self-reliance and preservation of local identities. Fertility patterns across regions coverage very slowly, which results in continuously increasing global population. Economic development is primarily regionally oriented and per-capita economic growth and technological change are more fragmented and slower than under other scenarios.

- The **B1 scenario family** describe a convergent world with the same global population that peaks in mid-century and declines thereafter, as in the A1 family, but with rapid changes in economic structures toward a service and information economy, with reductions in material intensity, and the introduction of clean and resource-efficient technologies. The emphasis is on global solutions to economic, social and environmental sustainability, including improved equity, but without additional climate initiatives.

- The **B2 scenario family** describe a world in which the emphasis is on local solutions to economic, social, and environmental sustainability. It is a world with continuously increasing global population at a rate lower than A2, intermediate levels of economic development, and a less rapid and more diverse technological change than under the B1 and A1 scenarios. While the scenario is also oriented toward environmental protection and social equity, it focuses on local and regional levels.

The linguistic nature of the scenario descriptions would suggest that underlying models are unlikely to be precisely defined and hence, may be more realistically described in terms of the uncertain models given in Definition 6. Also, the variation amongst related scenarios [8] provides further justification for thinking of scenarios as fuzzy constructs. Within each family there tends to be significant variation in the range of CO_2 emissions trajectories. It is not, however, appropriate to apply any statistical interpretation to the frequency or spread of trajectories within a family. These are not samples from a random process. Instead they are generated by different models based on different assumptions that correspond, to a greater or lesser extent, to the linguistic definition of the scenario. Whilst resisting any frequentist interpretation of the published family of emissions time series we can nonetheless suppose that there are regions within the bounds of the family of time series that are thought of as being more representative of the scenario, so should have greater membership in the scenario. A specific 'marker scenario' is taken as being representative of the family so should occupy the region of maximum fuzzy membership in the scenario.

Figure 6 shows a fuzzy set representing the A1 family of scenarios [8]. The fuzzy set was generated by taking the outer envelope of the published family of scenarios to represent the extreme bounds on the fuzzy set . We have then generated a trapezoidal fuzzy set based on these outer bounds and an interval around the maker scenario. The fuzzy set need not be trapezoidal and in fact sensitivity analysis indicates that any membership function consistent with the general form of the scenario family will generate similar results.

In the current study we have used the spatially aggregated climate model MAGICC [14]. Fuzzy emissions scenarios were constructed using the method outlined above for the A1, A2, B1 and B2 families of scenarios (see [8]) and propagated through MAGICC using default values for the model parameters according to a standard α-cut method as follows: A fuzzy CO_2 emissions scenario (as shown in Figure 6) defines a possibility distribution π^t for each year t. The corresponding fuzzy projection of global mean temperature (GMT) change at time t' is then taken to have possibility distribution $\pi^{t'}$ such that:

$$\forall \alpha \in [0,1] \;\; \pi_\alpha^{t'} = \left\{ y_{t'} : \pi^{t'}\left(y_{t'}\right) \geq \alpha \right\} = \left[\underline{y_{t'}}, \overline{y_{t'}}\right]$$

where $\underline{y_{t'}}$ ($\overline{y_{t'}}$) is the lower (upper) bound on GMT change at t' obtained from MAGICC as the input CO_2 emissions at t (x_t) is varied across the range π_α^t. The possibility distribution is then recovered from the α-cuts in the normal manner by taking:

$$\pi^{t'}\left(y_{t'}\right) = sup\left\{ \alpha : y_{t'} \in \pi_\alpha^{t'} \right\}$$

Although, in practice, the α-cuts were only evaluated at 6 levels, as shown in Figure 7, and the remaining α-cuts estimated by interpolation. The calculation of upper and lower bounds on GMT change for each α-cut was simplified by taking advantage of the monotonicity of MAGICC with respect to fossil CO_2

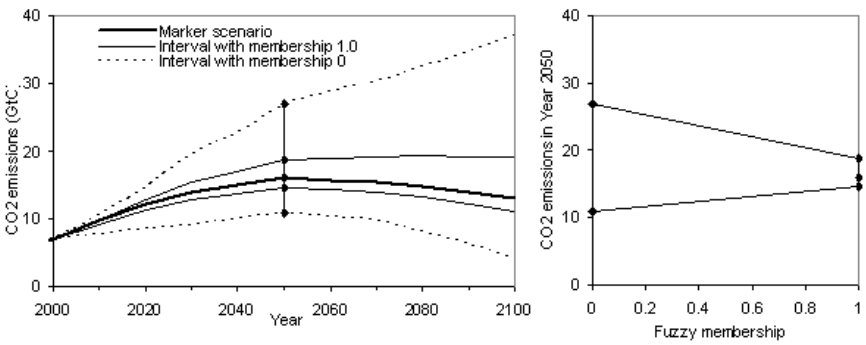

Figure 6: Fuzzy set representing CO_2 emissions for the $A1$ family of scenarios.

emissions. More details concerning the propagation of fuzzy scenarios through MAGICC can be found in [7].

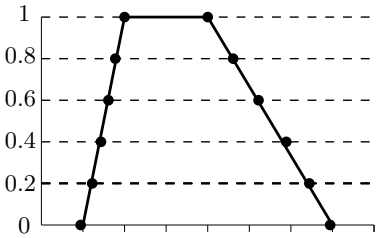

Figure 7: α-cuts of a fuzzy set on CO_2 emissions

The resulting fuzzy global mean temperature projections are shown in Figure 8 for the year 2100.

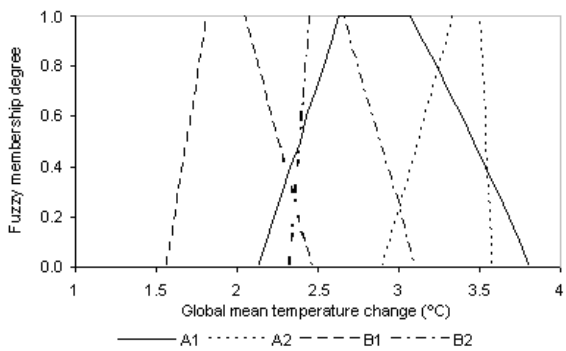

Figure 8: Fuzzy global mean temperature projections for 2100

To combine the different fuzzy projections it is proposed to define a fuzzy measure on the power set of scenarios $D = \{A1, A2, B1, B2\}$ and then to aggregate the different membership values resulting from the four scenarios using the Choquet integral. The current view that emerged during interviews with economists from the Tyndall Centre was that the likelihood of the 4 scenarios follows the ordering $\mu(A1) > \mu(B1) > \mu(A2) > \mu(B2)$. Figure 9 shows fuzzy measures in the same proportions, preserving this ordering, whilst assuming different degrees of synergy or redundancy as represented by sub-additive, additive and super-additive measures. The resulting aggregated fuzzy sets on global mean temperature for 2100 based on these measures and combined using the Choquet integral are shown in Figure 10.

Subset of scenarios	Additive	Sub-additive	Super-additive
$\{A1\}$	0.5	0.7	0.3
$\{A2\}$	0.2	0.28	0.12
$\{B1\}$	0.25	0.35	0.15
$\{B2\}$	0.05	0.07	0.03
$\{A1, A2\}$	0.7	0.74	0.63
$\{A1, B1\}$	0.75	0.79	0.68
$\{A1, B2\}$	0.55	0.58	0.5
$\{A2, B1\}$	0.45	0.47	0.41
$\{A2, B2\}$	0.25	0.26	0.23
$\{B1, B2\}$	0.3	0.32	0.27
$\{A1, A2, B1\}$	0.95	0.93	0.91
$\{A1, A2, B2\}$	0.75	0.74	0.72
$\{A1, B1, B2\}$	0.8	0.78	0.77
$\{A2, B1, B2\}$	0.5	0.49	0.48
$\{A1, A2, B1, B2\}$	1	1	1

Figure 9: Table showing fuzzy measures applied to scenarios

5 Summary and Conclusions

A granular interpretation of fuzzy measures has been proposed whereby the fuzzy measure of a set of propositions represents an approximation to the probability of the disjunction of these propositions. This approx-

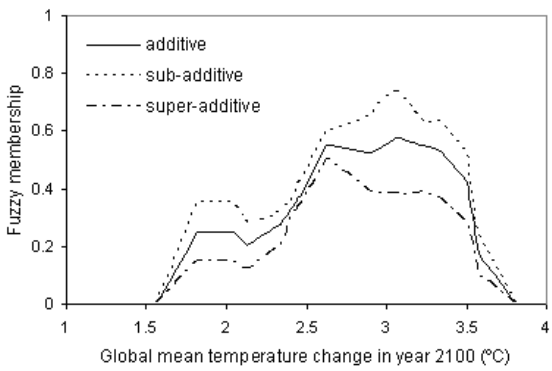

Figure 10: Aggregated fuzzy global mean temperature projections for 2100

imation is based on known probability values across a relatively coarse partition of the set of possible worlds. In the case that the meaning of propositions are uncertain then this semantics can be extended to incorporate random set based models.

The granular semantics outlined in this volume was motivated by an investigation into applying fuzzy measures to quantify the uncertainty associated with climate emission scenarios. These scenarios can be viewed as identifying sets of possible worlds where each world corresponds to a complex socio-economic state. Scenarios are based on linguistic storylines and hence are naturally imprecise constructs. In our study fuzzy measures were applied in conjunction with the Choquet integral in order to aggregate fuzzy global mean temperature predictions resulting from the propagation of fuzzy CO_2 levels through the climate model MAGICC.

The delicate problem of elicitation remains - the work described above was based on typical values rather than a formal elicitation exercise (although informal interviews were conducted). However, we have argued that the above granular semantics for fuzzy measures defined across the possible states of the world will provide a convenient framework for the elicitation process.

6 Acknowledgements

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