LIKELIHOOD-BASED STATISTICAL DECISIONS

Marco Cattaneo
Seminar for Statistics
ETH Zürich, Switzerland

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Let $\mathcal{P}$ be a **set of statistical models**:

- $\mathcal{P}$ is a set of probability measures on a measurable space $(\Omega, \mathcal{A})$;
- absolutely no structure is imposed on $\mathcal{P}$ (for instance, $\mathcal{P}$ could be a nonparametric family of models).

The **likelihood function** $lik_A : \mathcal{P} \to [0, 1]$ based on the observation $A \in \mathcal{A}$:

- is defined by $lik_A(P) = P(A)$;
- measures the relative plausibility of the models $P \in \mathcal{P}$, on the basis of the observation A alone;
- is not calibrated: only ratios $lik_A(P)/lik_A(P')$ are well determined in a statistical sense.

Relative Plausibility

The **relative plausibility** rp on $\mathcal{P}$ generated by lik_A:

- is the class of nonnegative functions on $2^{\mathcal{P}}$ defined by $rp(\mathcal{H}) \propto \sup_{P \in \mathcal{H}} lik_A(P)$;
- is a non-calibrated possibility measure on $\mathcal{P}$.

The **description of the uncertain knowledge about the models** by means of relative plausibility:

- can be easily updated (since $lik_{A \cap B} = lik_A \, lik_{B|A}$);
- allows a natural incorporation of prior information: independent pieces of information can be combined, and complete (or partial) ignorance can be described;
- is parametrization invariant;
- satisfies the strong likelihood principle; but can also use pseudo likelihood functions;
- can be used for inference (maximum likelihood estimator, tests and confidence regions based on the likelihood ratio statistic, . . .) and decision making (MPL criterion);
- leads to conclusions which are in general weaker than those based on a probabilistic description of the uncertain knowledge about the models; but is based on weaker assumptions, is simpler and more intuitive.

MPL Criterion

A **statistical decision problem** is described by a loss function $L : \mathcal{P} \times \mathcal{D} \to [0, \infty)$:

- $\mathcal{D}$ is the set of possible decisions, and $\mathcal{P}$ is the set of considered statistical models;
- $L(P, d)$ is the loss we would incur, according to the model P, by making the decision d.

If the uncertain knowledge about the models is described by the relative plausibility rp on $\mathcal{P}$, the **MPL criterion** for choosing a decision $d \in \mathcal{D}$ consists in minimizing $\sup_{P \in \mathcal{P}} rp\{P\}\, L(P, d)$:

- the minimized quantity is the Shilkret integral of $L(\cdot, d)$ with respect to rp: this is intuitive and simple (allowing decisions even in difficult problems);
- if invariance with respect to translations of the loss function is needed, the integral of Choquet should be used instead of the one of Shilkret;
- the obtained decision functions are equivariant (if the problem is invariant) and asymptotic optimal (if some regularity conditions are satisfied);
- the consideration of many examples suggests that the MPL criterion leads in general to reasonable decisions.

Example

Estimation of the variance components in the 3×3 random effect one-way layout, under normality assumptions and weighted squared error loss.

$$X_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad \forall\, i, j \in \{1, 2, 3\}$$

Normality assumptions:

$$\alpha_i \sim \mathcal{N}(0, v_a), \;\; \varepsilon_{ij} \sim \mathcal{N}(0, v_e), \;\; \text{all independent}$$

$\Rightarrow\;\; X_{ij} \sim \mathcal{N}(\mu, v_a + v_e)$ dependent, $\mu \in \mathbb{R}$, $v_a, v_e \in \mathbb{R}^+$

The estimates $\widehat{v_e}$ and $\widehat{v_a}$ of the variance components v_e and v_a are functions of

$$SS_e = \sum_{i=1}^{3} \sum_{j=1}^{3} (x_{ij} - \bar{x}_{i\cdot})^2 \quad \text{and} \quad SS_a = 3 \sum_{i=1}^{3} (\bar{x}_{i\cdot} - \bar{x}_{\cdot\cdot})^2 \; ,$$

where

$$\bar{x}_{i\cdot} = \frac{1}{3} \sum_{j=1}^{3} x_{ij} \; , \quad \bar{x}_{\cdot\cdot} = \frac{1}{9} \sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \; ,$$

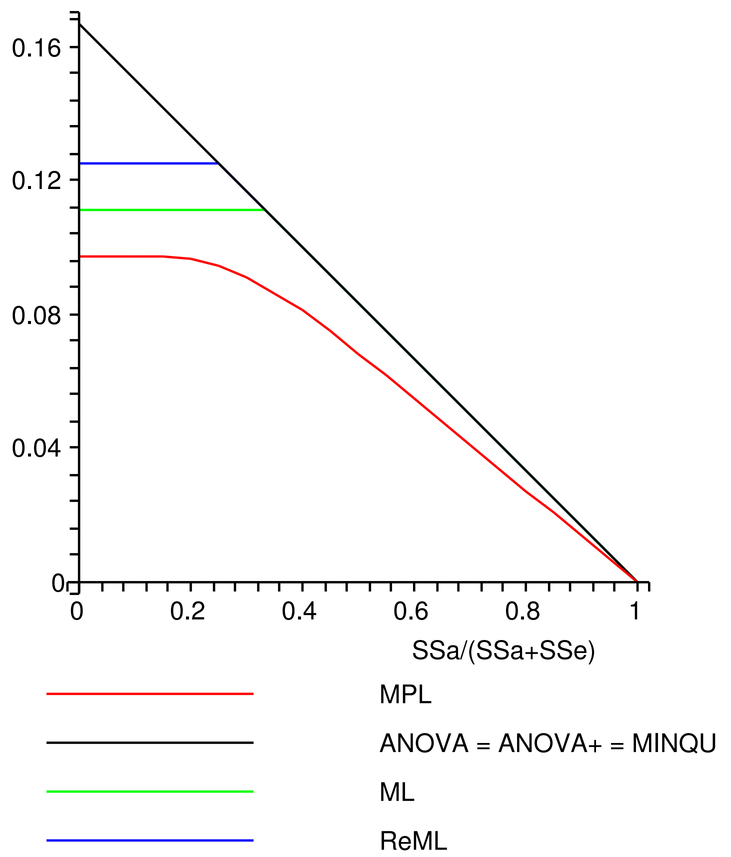
$$\frac{SS_e}{v_e} \sim \chi^2_6 \quad \text{and} \quad \frac{\frac{1}{3}\, SS_a}{v_a + \frac{1}{3}\, v_e} \sim \chi^2_2 \; .$$

The considered loss functions are

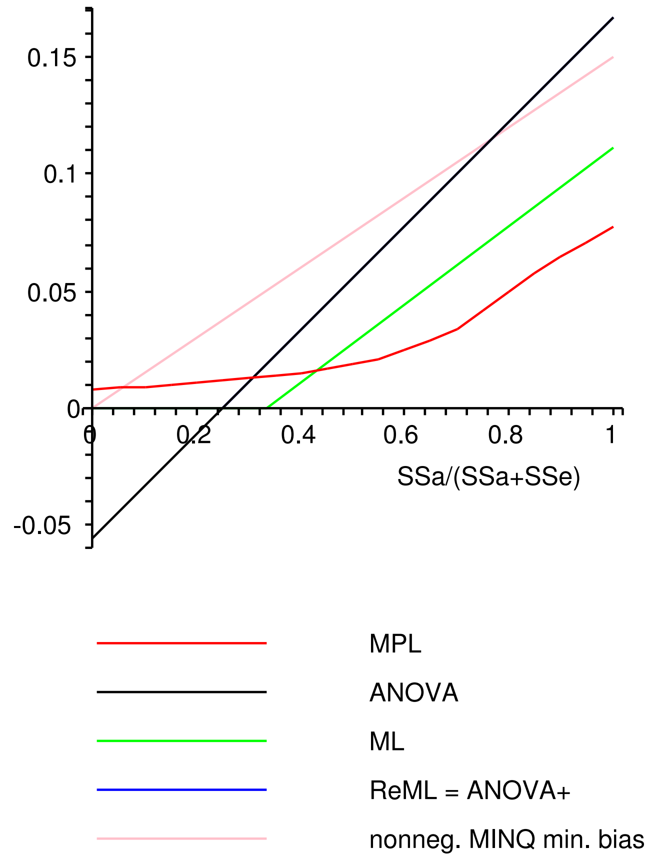
$$3\, \frac{(\widehat{v_e} - v_e)^2}{{v_e}^2} \quad \text{and} \quad \frac{(\widehat{v_a} - v_a)^2}{(v_a + \frac{1}{3}\, v_e)^2} \; .$$

Example

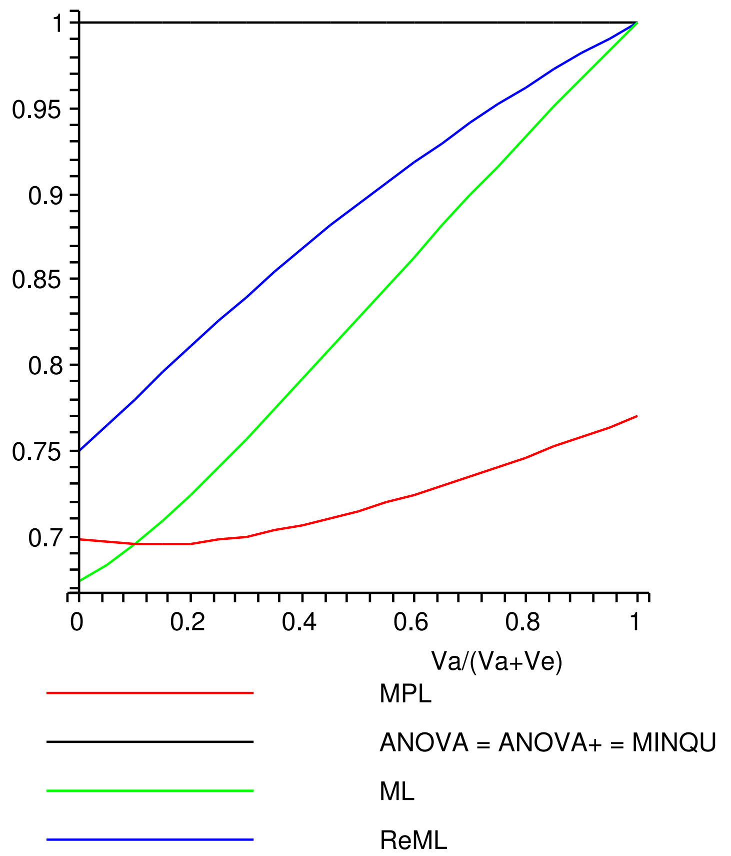
$$\frac{\widehat{v_e}}{(SS_a+SS_e)}$$



$$\frac{\widehat{v_a}}{(SS_a+SS_e)}$$



$$3\,\frac{E[(\widehat{v_e}-v_e)^2]}{{v_e}^2}$$



$$\frac{E[(\widehat{v_a}-v_a)^2]}{(v_a+\frac{1}{3}\,v_e)^2}$$